PARTIAL CONFOUNDING IN FACTORIAL EXPERIMENTS

by

Sate Songhorabadi

Thesis submitted for

The degree of

Master of Science

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF SOUTHAMPTON

OCTOBER 1979

ACKNOWLEDGEMENTS

I would like to express my appreciation to my supervisor Dr. J.A. John for his continuous assistance throughout the course of this work. I would also like to express my debt to Dr. S.M. Lewis for her help when the computer program was being developed.

SUMMARY

In situations where the experimenter demands the study of behaviour of a few factors at the same time and the way these factors may interact a form of experimental design is adopted known as factorial experiments.

Due to practical constraints of running an experiment often providing homogeneous conditions for all treatments is impossible and hence smaller blocks than a full replication of the treatments are used. Use of small blocks cause introduction of certain amounts of variability to the responser due to heterogenous conditions for the blocks. This by itself gives way to a loss of accuracy of certain treatment comparisons in the analysis part.

Previously, work has been done in use of complete and incomplete block designs in conjunction with factorial experiments, so that the loss in the accuracy of main effects and low order interactions is low.

In this work generalised cyclic designs are introduced and work on their properties and mathematical models for their use in conjunction with factorial experiments is reviewed.

Criterion is introduced for distinguishing designs of given size, so that the most suitable forms of allocating treatments to blocks can be obtained. This criterion is known as efficiency.

Further, methods of obtaining good designs and past conjectures are discussed.

Finally, tables of cyclic designs and generalised cyclic designs with their efficiency values are given.

Tables of cyclic designs contain information on full sets of cyclic designs, within the given range.

Tables of generalised cyclic designs of order two contains the most efficient designs with respect to main effects and first order interactions among a large number of designs that were run on the EFFICIENCY program.

TABLE OF CONTENTS

1. INTRODUCTION

Factorial designs are a useful class of design by which the interests of the experimenter demands that effects of different factors should be studied simultaniously. This can be done by varying the treatments and testing the treatments under a series of differing conditions. Situations where such designs are useful is when effects of a few factors have to be determined over a specified range and/or it is required to investigate interactions (interrelations) between effects of several factors. Some of these factors may be of special interest, while others are merely recognizable nuisances. The original philosophy and practice of factorial experiments was mainly due to work of R.A. Fisher and his book "The design of experiments" published in 1935.

There are several important advantages involved in adopting a factorial design rather than the classical arrangement of varying only one factor at a time. First of all, one obtains a much more general picture of the effect of each factor, because of the range of conditions provided by variations in the other factors. Secondly, the variety of factor combinations within the experiment facilitates the prediction of results likely to be achieved under a particular combination of circumstances. Thirdly, if the factors included in the experiment have statistically independent effects then as much information is available about each factor as would be obtained if the whole experiment dealt with only one factor, the rest remaining constant. Fourthly, as often happens, the factors are not independent, but have effects that are to a greater or lesser degree correlated. In this situation only a factorial experiment can provide information about the nature of these interactions. This property of the factorial experiments will be used to a great extent later on in this project.

1.1 INTERACTIONS AND MAIN EFFECTS

One concept introduced above was the introduction of the term interaction, we start by introducing the mathematical model. Suppose we are trying to examine the effects of just two factors A and B, then we conjecture that observed effect on some experimental unit of the ith level of A and jth level of B can be written in the form

$$y_{ijk} = \mu + a_i + b_j + c_{ij} + e_{ijk} \tag{1}$$

	y_{ijk},	is the observed measurement
	μ ,	is an overall average
	a_i, b_j	are relative contributions of the two factors
	c_{ij}	is the interaction contribution
and	e_{ijk}	is the random deviation imposed.

In the analysis part we study the significance of a,b,c terms and results are represented in the well known analysis of variance tables.

The affect of a significant interaction can be represented in the following figures, for $i=1,2$, $j=1,2,3$.

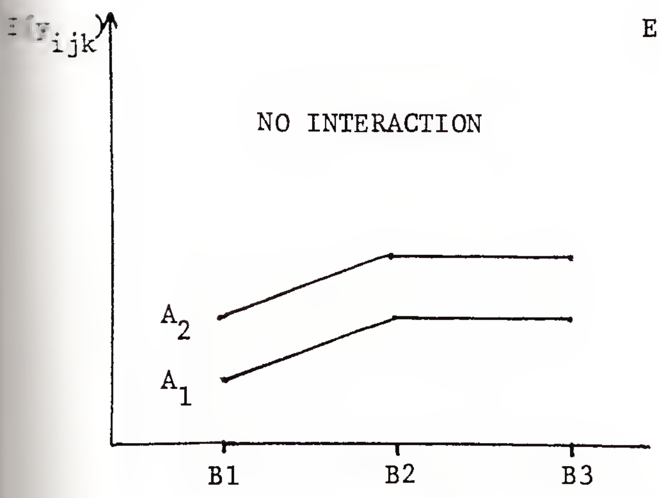


FIGURE 1.

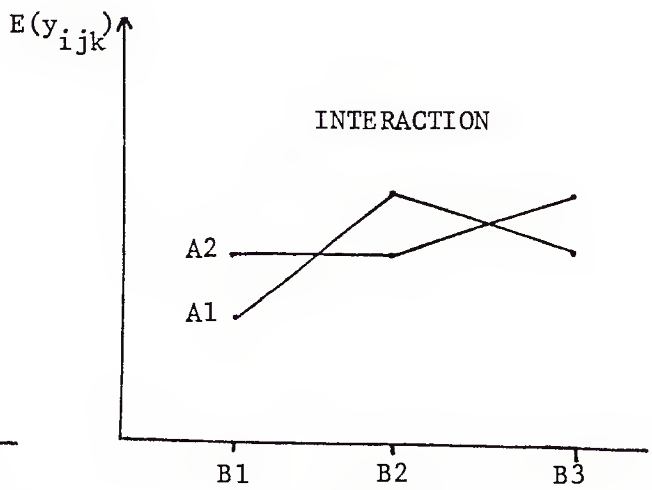


FIGURE 2.

The difference between the two figures is that in the case of No Interaction the observed measurements y_{ijk} give parallel lines, while in Figure 2 the lines are not parallel. That is the difference between each level of B does not change in Figure 1, no matter what level of A is involved. In Figure 2 lack of parallelism indicates that apart from main effects some additional effects are taking place. This additional effect is known as the interaction between the two main effects.

The above models were only for two main effects present. When it is known that several relevant factors are present, the case for some kind of factorial design is overwhelming. However In situations where a large number of factors are involved the magnitude and complexity of the task of design becomes a major problem, and often makes a homogenous treatment of a full replication of the experiment impossible. Hence we are forced into making a decision between involving a large number of treatments or reducing the "efficiency" of the design.

From the elementary form of the model (1) it is possible to develop models involving large numbers of factors, block, interactions and other sophisticated plans required by practical constraints.

1.2 FACTORIAL IN COMPLETE BLOCK DESIGNS

Most types of experimental plans are suitable to be used in conjunction with factorial experiments. If the number of treatment combinations is not large we can have factorial design in complete block designs such as randomized blocks and Latin squares. Complete blocks are used with their usual advantages, in such a way as to remove effects brought into the experiment due to varying conditions of the

experiment. With randomized block designs the grouping of experimental units into blocks are used to reduce the error of an experiment in one direction only. With the Latin square designs the grouping of replications in two directions is employed.

1.3 FACTORIAL IN INCOMPLETE BLOCK DESIGNS

For large experiments the major source of error in lack of homogenous replications, and the common method of overcoming the problem is use of a block which is smaller than a complete replication. Class of designs involving block sizes less than a complete replication are called incomplete block designs. These designs were introduced by F. Yates (1936). In comparison to complete block designs described earlier, heterogeneity is eliminated to a greater extent, but the reduction in size of a block is achieved by sacrificing all or part of the information on certain treatments comparisons.

1.4 CONFOUNDING

In factorial experiments use of such blocking will cause the sacrifice of accuracy on certain treatment effects, in practice for most designs high order interactions are assumed negligible and hence main effects and first order interaction comparisons are of prime importance. This property can be used so that accuracy of high order interactions are sacrificed.

Using a small block and hence sacrificing high order interaction is known as confounding. It will allow precise comparison of main effects and important interactions to be made without the involvement of block effects.

It is often possible to construct confounded designs in a few replicates in a way that different replicates will cause different

treatment effects to be confounded, this is known as partial confounding. Partial confounding allows some information to be retained on all effects and block to block variation is eliminated at the same time.

It is obvious that the best designs are always designs that allow a full replication of the treatments to be made in one block.

As will become clear later we are only concerned with multireplicate designs that is designs that will have partial confounding.

1.5 BALANCED INCOMPLETE BLOCK DESIGNS

One special form of incomplete block designs are balanced incomplete block designs, these designs ensure that all pairs of treatments are compared with the same precision, this can be achieved by retaining balance in the design so that every pair of treatments occur together in the same number of blocks and hence all treatment comparisons are of equal accuracy. The problem is that these designs only exist for a certain number of treatments, block size and number of replications. In situations where there are no balanced designs further sacrifice of accuracy should be made such that certain treatment contrasts have lower accuracy than others.

1.6 PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

As an extension to balanced incomplete block designs, these are a group of designs known as partially balanced incomplete block designs represented by (PBIB/m), by which, rather than every treatment occuring the same number of times, there are m associate classes, and hence treatment comparisons can be estimated with m degrees of accuracy. As an example group divisible designs are a large and important class of

partially balanced incomplete block designs with 2 associate classes (PBIB/2). t treatments may be divided into m_1 groups each containing m_2 treatments, where $t=m_1m_2$. Any two treatments in the same group are first associates and two treatments in different groups are second associates.

1.7 SOME REFERENCES ON FACTORIAL EXPERIMENTS IN (PBIB/2)

In the class of partially balanced incomplete block designs with two associate classes Kramer and Bradley (1957) and Zelan (1958) considered group divisible designs and Bradley, Walpole and Kramer (1960) considered Latin square designs with two association schemes. Brenna and Kramer (1961) considered the use of factorials in rectangular lattice designs. Kurkjian and Zelan (1962,1963) introduced a special calculus for the analysis factorial experiments and gave a class of designs having an orthogonal factorial structure for two factor experiments, and John (1973) has considered use of cyclic designs.

1.8 CYCLIC DESIGNS

Cyclic designs are a group of designs belonging to a class of incomplete block designs. As the name suggest by the cyclic development of the initial block(s) it is possible to obtain all other blocks of the design.

Bose and Nair (1939) have suggested by making the right choice of initial block and its cyclic development it is possible to construct designs with high degree of balance giving balanced incomplete block designs and partially balanced incomplete block designs.

Cyclic designs as a class in their own right were introduced for blocks of size 2 by Kempthorne (1953), Zoellner and Kempthorne (1954) and McKeon (1960). Designs for block size 2 were more fully studied

in two papers by David (1963) and (1965). In these papers it was shown how non isomorphic designs of a given size can be constructed. These results were extended to general block sizes by Wolock (1964), David and Wolock (1965), John (1966).

Altogether up till now all balanced and partially balanced designs considered do not have the necessary flexibility required by the factorial experiment. Factorial experiments are multifactor experiments where the number of factors involved can be high, and in one experiment we are concerned with main effects as well as the interactions.

In the case of balanced data, we perform an analysis of the orthogonal treatment combinations.

In the case of non orthogonal block designs it is possible to perform an orthogonal analysis of the main effects and interactions provided certain conditions are present. These conditions were shown by Cotter, John and Smith (1973). It is difficult to explain this result at this stage without the necessary mathematics. However, it requires that the design can be represented by a matrix A, that has a block circulant property.

That is for an n factor design with levels $m_1, m_2, \ldots m_n$

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_{m_1} \\ A_{m_1} & A_1 & & \\ \vdots & & \ddots & \\ A_2 & & & A_1 \end{bmatrix}$$

and

$$A_{i} = \begin{bmatrix} A_{1i} & A_{2i} & \cdots & A_{m_2 i} \\ A_{m_2 i} & \ddots & & \\ \vdots & & \ddots & \\ A_{2i} & & & A_{1i} \end{bmatrix}$$

for $i=1,\ldots,m_1$

and so on for $A_{ijk}\ldots$ such that the last level of blocking has a square matrix of order m_n with row and column sums all equal.

Designs that have this property are said to have a factorial structure. Cyclic designs mentioned previously have a property that every n cyclic designs of size k can be combined to give a generalised cyclic design of order n represented by GC/n. John (1973) has shown that this generalised cyclic design have a factorial structure.

Later on in the thesis when the mathematical theory is more developed results of Cotter, John and Smith (1973) and John (1973) will be shown. However at this stage these results were given only to show the relation between cyclic designs, generalised cyclic designs and factorial structures.

2.1 GENERALISED CYCLIC DESIGNS (GC/n)

Generalised cyclic designs belong to the class of incomplete block designs. For an n factor generalised design we have an n tuple $a_1 \ldots a_n$ set, representing a particular treatment.

a_i is an integer from o to $m_i - 1$ where m_i is the number of levels associated with the factor i $(i=1,\ldots,n)$.

EXAMPLE

We have two factors a_1 and a_2 each at 3 and 2 levels respectively. Then levels of a_1 is represented by 0,1 or 2 and a_2 is represented by 0 or 1. The complete set of all the possible treatments of this 3 by 2 design is

00, 01, 10, 11, 20, 21

This ordering of treatments in such a way that the right most digit in varying fastest is called LEXICOGRAPHICAL ordering. Now for a block size of $k=3$ we can have (00, 10, 21) as one possible block having three of the treatments occurring only once. It is clear that considering the first digits we have a cyclic design (GC/1) for the factor a_1 that is (0, 1, 2), and considering second digits we have a cyclic design for a_2 that is (0, 0, 1).

By cyclic development of the above 2nd order generalised cyclic design we obtain the full set of the design with 6 blocks.

(00, 10, 21)	(01, 11, 20)
(10, 20, 01)	(11, 21, 00)
(20, 00, 11)	(21, 01, 10)

Cyclic development is done in such a way that on columns we cycle a_1 and on rows we cycle a_2 .

The total number of treatments is 6 which is given by $m_1 \times m_2$. For a general case with n factors the total number of treatments in

$$t = m_1 m_2 \dots m_n$$

We follow the same representation of the parameters of the GC/n design as the one used by John (1973).

$(t,k,r;\ m_1,\dots,m_n)$ where
- t = total number of treatments
- m_i = number of levels associated with factor a_i
- n = number of factors
- k = block size
- r = number of replications.

The 3×2 design with initial block (00, 10, 21) is therefore represented by (6, 3, 3; 3, 2).

It is clear that there may be many initial blocks with their own full set of blocks that have the above representation of the parameters. However they may not be of equal interest as far as comparison of main effect and interactions are concerned.

All treatments in the above design occur an equal number of times, but we note that the number of times two treatments occur together in the same block is not the same. 00 and 11 occur together in the same block twice but 00 and 01 do not occur together in any of the blocks.

The manner in which two treatments occur together in blocks is if great importance particularly later when we consider methods by which one initial block is to be preferred to another.

2.2 PROPERTIES OF GENERALISED CYCLIC DESIGNS

(A) Nonisomorphic sets: Other members of a set generated by a particular initial block have the same properties as far as accuracy

of comparison of main effects and interactions are concerned. In such situations what actually is taking place is that by generating the other blocks of an initial block we are relabelling the treatments differently. Among these blocks only one is sufficient to give all the necessary information we require on the design. In these cases we obtain the block giving the lowest numerical value among all equivalent blocks so that we have a unique representation of the designs and hence obtain a group of non-isomorphic sets.

We define an operator $R(h_1 \ldots h_n)$ such that operation on an n-tuple treatment $a_1 \ldots a_n$ changes it to an n tuple treatment $b_1 \ldots b_n$.

$$(a_1 \ldots a_n)R(h_1 \ldots h_n) = b_1 \ldots b_n \tag{2}$$

or

$$a_1 R(h_1) \; . \; . \; . \; a_n R(h_n) = b_1 \ldots b_n \tag{3}$$

where h_i is a non zero number less than m_i relatively prime to m_i. $R(h_1 \ldots h_n)$ can take many different values and operation on initial block $(a_1 \ldots a_n)$ can produce the other blocks of an equivalent class. However it must be noted that many of the blocks produced by this operation may not even belong to the full set of $(a_1 \ldots a_n)$ and so they can produce other blocks by a cyclic rotation of the treatments. The way in which h_i produces a relabelling of a_i is as follows, h_i is multiplied by a_i and then mod m_i is applied when $h_i \times a_i$ is greater than m_i.

Consider a set of sizes (12, 4, 4; 4, 3) as defined above. All possible values for $R(h_1,h_2)$ are

$$R(1,2),\ R(3,1),\ R(3,2)$$

then

$$(00,01,11,20)R(3,1) \equiv (00,01,31,20)$$

Applying cyclic rotations on either of these two equivalent blocks we obtain 12, corresponding equivalent blocks. It can be shown that by

the above $R(h_1,h_2)$ operations and cyclic developments, all the following blocks are equivalent.

(00,01,11,20), (00,01,20,31), (00,01,10,21), (00,01,21,30) .

Block (00,01,10,21) has the lowest numerical values and is therefore the initial block.

If the elimination of the equivalent designs are performed systematically it is a very efficient method of reducing the total number of blocks to a few intial blocks.

(B) Fractional sets:

Consider a set of size (16,4,4; 4,4) with the initial block

(00 11 22 33)

Performing cyclic rotations on the two factors we obtain

(00 11 22 33)	(10 21 32) 03)	(20 31 02 13)	(30 01 12 23)
(01 12 23 30)	(11 22 33 00)	(21 32 03 10)	(31 02 13 20)
(02 13 20 31)	(12 23 30 01)	(22 33 00 11)	(32 03 10 21)
(03 10 21 32)	(13 20 31 02)	(23 30 01 12)	(33 00 11 22)

All the 4 blocks in any row or column are identical with 4 blocks in any other row or column. Therefore the full set is not composed of 16 blocks as may be expected but only 4 unique blocks. We refer to this set as a 1/4 fractional set.

If t and m_i have a common devisor p_i for each particular i then a fractional set of size $(t,k,k/p; m_1,\ldots,m_n)$ exists for p where $p = p_1p_2\ldots p_n$.

In the above example $P=4=2.2$ and so the fractional set is of size (16,4,1; 4,4). The full set is equivalent to replicating

the fractional set 4 times. For every fractional set of size $(t,k,k/p;\ m_1 \ldots m_n)$, there exists an equivalent full set represented by $(t/p,k/p,k/p;\ \frac{m_i}{p_1}, \ldots, \frac{m_n}{p_n})$.

In the example full set is of size (4,1,1;2,2) given by (00), (01), (10), (11).

Generalised cyclic designs by their high degree of balance and also by the way they are able to restrict the number of possible permutations of treatments in a block (Property A) are attractive designs for situations where a number of treatments exceeds the maximum sensible value for a block size. They are very flexible designs and due to property B it is always possible to form some kind of design for given values of t,k,r and m_i .

Ease of representation is another advantage of these designs. The initial block contains all the necessary information for a design of given size, the rest of the design can be obtained from the circulant property of the design. This is a major advantage for analysis or situations where a comparative study of designs are involved.

Any GC/n design can be easily extended to GC/n+1 designs this dimensional property of generalised cyclic designs makes them particularly suitable to be used in conjunction with factorial and split plot designs, where some interactions and main effects are more important than others.

2.3 LINEAR MODELS FOR GENERALISED CYCLIC DESIGNS

We have assumed that all treatments within a block are performed under similar conditions, this follows that when an observation is made on a particular treatment certain amounts of variability will be due to block effects.

Model for representing the above block variation is as follows

$$y_{ij} = u + t_i + b_j + e_{ij} \qquad (4)$$

$$(i=1,=,\ldots t,\ j=1,2,\ldots k)$$

where y_{ij} are the observations

u is the general mean

t_i's are the treatment effect

b_j's are the block effects

and e_{ij}'s are the uncorrelated random deviations with mean zero and constant variance S^2 .

Alternatively (4) can be written in matrix form

$$y = (1\ X_t\ X_b)(u\ t\ b)' + e \qquad (5)$$

t and b are vectors of elements t_i and b_i respectively, and $(1\ X_t\ X_b)$ is the design matrix.

X_t and X_b are matrices representing presence or absence of a particular treatment effect or block effect.

X_t and X_b have the following properties.

(1) Summing rows of X_t or X_b give number of times treatments occur in design or number of units in each block respectively, that is vectors r and K .

(2) $X_t X_b'$ is called the incidence matrix N with elements n_{ij} n_{ij} is the number of times treatment i occurs in block j .

(3) Where y is the vector of observations.

$X_t'\ y = T$ vector of treatment totals.

$X_b'\ y = B$ vector of block totals.

$1'y = G$ Grand total.

Now

$$(1 \quad X_t \quad X_b)'(1 \quad X_t \quad X_b) = \begin{vmatrix} 1'1 & 1X_t' & 1X_b' \\ X_t'1 & X_tX_t & X_tX_b' \\ X_b'1 & X_bX_t' & X_bX_b \end{vmatrix} \tag{6}$$

$$\begin{vmatrix} n & r' & K' \\ r & r^d & N \\ K & N' & K^d \end{vmatrix}$$

Vectors K and r are row sums of X_b and X_t. Subscript d represents diagonal representation of a vector.

$$(1 \quad X_t \quad X_b)'(y) = (G \quad T \quad B)' \tag{7}$$

In terms of the usual normal equations $X'y = X'X$ we write

$$(1 \quad X_t \quad X_b)'(y) = (1 \quad X_t \quad X_b)'(1 \quad X_t \quad X_b) \tag{8}$$

giving

$$n\hat{u} + r'\hat{t} + K'\hat{b} = G \tag{9}$$

$$r\hat{u} + r^d\hat{t} + N\hat{b} = T \tag{10}$$

$$K\hat{u} + u'\hat{t} + K^d\hat{b} = B \tag{11}$$

Eliminating block effects $\hat{b}$ from the equation, by calculating

$$(10) - NK^{-d}\,(11)$$

we obtain

$$(r\text{-}NK^{-d}K)\hat{u} + (r^d\text{-}NK^{-d}N')\hat{t} + (N\text{-}NK^{-d}K^d)\hat{b} = T - NK^{-d}B \tag{12}$$

since $(r\text{-}NK^{-d}K) = 0$.

We have

$$(r^d\text{-}NK^{-d}N')(\hat{t}) = T - NK^{-d}B \tag{13}$$

Let $\quad A = (r^d\text{-}NK^{-d}N')$, matrix representing design

and $\quad Q = T - NK^{-d}B$, vector of adjusted treatment totals

giving

$$A \hat{t} = Q \tag{14}$$

(14) is the reduced normal equation. We let $W = (A+J)^{-1}$ be a generalised inverse of A where J is a $t \times t$ matrix of 1's.

Then

$$\hat{t} = WQ \tag{15}$$

Due to special structure of the cyclic designs we have equal block sizes and equal replications of treatments.

This makes r^d and K^d diagonal matrices such that constant r and K are multiplied by the identity matrix.

This simplifies A and Q as follows

$$A = rI - (1/K)\, NN' \tag{16}$$

$$Q = T - (1/K)\, NB \tag{17}$$

Furhter for generalised cyclic designs we can relate structure of NN' to structure of A.

EXAMPLE

Design of size (12,4,4;4,3) with initial block (00,11,21,32) is

(00, 11, 21, 32)	(01, 12, 22, 30)	(02, 10, 20, 31)
(10, 21, 31, 02)	(11, 22, 32, 00)	(12, 20, 30, 01)
(20, 31, 01, 12)	(21, 32, 02, 10)	(22, 30, 00, 11)
(30, 01, 11, 22)	(31, 02, 12, 20)	(32, 00, 10, 21)

$$
NN' = \begin{array}{c} \\ 00 \\ 01 \\ 02 \\ 10 \\ 11 \\ 12 \\ 20 \\ 21 \\ 22 \\ 30 \\ 31 \\ 32 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} 00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 & 30 & 31 & 32 \end{array} \\
\left[\begin{array}{ccc|ccc|ccc|ccc}
4 & 0 & 0 & 1 & 3 & 0 & 0 & 2 & 2 & 1 & 0 & 3 \\
0 & 4 & 0 & 0 & 1 & 3 & 2 & 0 & 2 & 3 & 1 & 0 \\
0 & 0 & 4 & 3 & 0 & 1 & 2 & 2 & 0 & 0 & 3 & 1 \\
1 & 0 & 3 & 4 & 0 & 0 & 1 & 3 & 0 & 0 & 2 & 2 \\
3 & 1 & 0 & 0 & 4 & 0 & 0 & 1 & 3 & 2 & 0 & 2 \\
0 & 3 & 1 & 0 & 0 & 4 & 3 & 0 & 1 & 2 & 2 & 0 \\
0 & 2 & 2 & 1 & 0 & 3 & 4 & 0 & 0 & 1 & 3 & 0 \\
2 & 0 & 2 & 3 & 1 & 0 & 0 & 4 & 0 & 0 & 1 & 3 \\
2 & 2 & 0 & 0 & 3 & 1 & 0 & 0 & 4 & 3 & 0 & 1 \\
1 & 3 & 0 & 0 & 2 & 2 & 1 & 0 & 3 & 4 & 0 & 0 \\
0 & 1 & 3 & 2 & 0 & 2 & 3 & 1 & 0 & 0 & 4 & 0 \\
3 & 0 & 1 & 2 & 2 & 0 & 0 & 3 & 1 & 0 & 0 & 4
\end{array}\right]
\end{array} \qquad (18)
$$

Elements of NN' in this particular case show the number of times each treatment occurs with other treatments. Due to this property NN' matrix is called the concurrence matrix. Also note that treatments are ordered in LEXICOGRAPHICAL ORDERING.

NN' has a block circulant structure which is similar to A, $(A+J)$ and $(A+J)^{-1}$. Hence by having one row or column of any of the above matrices it is possible to generate the whole matrix.

We define $R^{i}_{m_j}$ to be a circulant matrix with first row having a 1 in column i and zeros elsewhere.

m_j represent size of the matrix.

c_{ij} is the value in the jth column of the ith block of NN'.

So that for (18) we have

$$c_{11} = 4 \quad c_{23} = 0 \quad c_{41} = 1$$

Then for the example (Recall 18)

$$NN' = \sum_{i=1}^{4} \sum_{j=1}^{3} C_{ij} (R_4^i \otimes R_3^j) \tag{19}$$

For a 3 dimensional generalised cyclic design we have

$$NN' = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \sum_{i_3=1}^{m_3} C_{ijk} (R_{m_1}^{i_1} \otimes R_{m_2}^{i_2} \otimes R_{m_3}^{i_3}) \tag{20}$$

For GC/n designs A is given by

$$A = \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} a_{i_1 i_2 i_3 \cdots i_n} \left(\prod_{j=1}^{n} \otimes R_{m_j}^{i_j} \right) \tag{21}$$

where

$$\prod_{j=1}^{n} \otimes R_{m_j}^{i_j} = R_{m_1}^{i_1} \otimes R_{m_2}^{i_2} \otimes \cdots \otimes R_{m_n}^{i_n}$$

Since GC/n designs are partially balanced incomplete block designs. It follows that W has the same pattern as A , (John 1973) namely

$$W = \sum_{i_1}^{m_1} \cdots \sum_{i_n}^{m_n} w_{i_1 \cdots i_n} \left(\prod_{j=1}^{n} \otimes R_{m_j}^{i_j} \right) . \tag{22}$$

3.1 EXTENSION OF GC/n DESIGNS TO FACTORIAL MODELS

So far in the considerations of the model building we have been concerned with structure of the blocks used.

Recall (4)

$$y_{ij} = u + t_i + b_j + e_{ij}$$

$$(i=1,2,\ldots t)$$

$$(j=1,\ldots k)$$

Now consider the treatment part of the model, that is the t_i's . As we mentioned before in factorial experiments we conjecture that value of every observed treatment can be partitioned into part due to its main effects, its 1st order interactions and so on for higher order interactions. Therefore the model can be written for $i=1\ldots t$ as

$$t_i = \sum_{p=1}^{n} a_p(i_p) + \sum_{p<q=1}^{n} a_{pq}(i_p,i_q) + \ldots + a_{12\ldots n}(i_1\ldots i_n) \qquad (23)$$

a_p refers to main effect of factor p and a_{pq} to interaction between factors p and q etc. $i_1\ldots i_n$ are levels of each factor corresponding to treatment combination i .

Recall (7)

$$(1\ X_t\ X_b)'(y) = (G\ T\ B)'$$

T is the vector of treatment totals. In lexicographical order we write T .

$$\begin{array}{llllll} T' = (T_{00\ldots0} & T_{0\ldots1} & T_{0\ldots2} & \cdots & T_{0\ldots(m_n-1)} & \\ \quad T_{00\ldots10} & \cdots & & & T_{0\ldots(m_n-1)} & \qquad (24) \\ \quad \vdots & & & & & \\ \quad T_{(m_1-1)(m_2-1)\ldots0} & & & & T_{(m_1-1)\ldots(m_n-1)} & \end{array}$$

Similarly we represent t as the vector of treatment effects

$$t' = (t_{0..0} \; t_{0..01} \quad \cdots \quad t_{(m_1-1)\ldots(m_n-1)})$$

Note the subscripts of t_i so far referred to decimal numbers $i=1\ldots t$ but now they refer to corresponding number between 1 to t, expressed by levels of the n tuple treatments.

EXAMPLE

For $n=2$ with $m_1=4$ and $m_2=3$, we write the vector of treatment effects.

$$t = (t_{00} \; t_{01} \; t_{02} \; t_{10} \; t_{11} \; t_{12} \; t_{21} \; t_{22} \; t_{30} \; t_{31} \; t_{32})$$

The average effect of first factor at the ith level is

$$t_{i.} = \frac{1}{3}\left[t_{i0} + t_{i1} + t_{i2}\right] \qquad \text{for } i=0,1,2$$

To test main effect of first factor, we test

$$H_0: \quad t_{0.} = t_{1.} = t_{2.}$$

By putting $\qquad t_{..} = \frac{1}{12}\sum_{i=0}^{2}\sum_{j=0}^{3} t_{ij} = \text{Overall mean}$

Equivalently H_0 can be expressed as

$$H_0 : \begin{cases} t_{0.} - t_{..} = 0 \\ t_{1.} - t_{..} = 0 \\ t_{2.} - t_{..} = 0 \end{cases}$$

In matrix notation the null hypothesis is

$$C^{10} \, t = 0$$

where

$$C^{10} = \frac{1}{12}\begin{pmatrix} 2 & 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \otimes (1 \quad 1 \quad 1 \quad 1)$$

$$= \frac{1}{12}(3I_3 - J_3) \otimes (1'_4)$$

$$= \frac{1}{12} C_3^1 \otimes C_4^0$$

We define
$$C_{m_j}^{k_j} = \begin{cases} m_j I_{m_j} - J_{m_j} & \text{if } x_j = 1 \\ 1'_{m_j} & \text{if } x_j = 0 \end{cases}$$

The subscript in C^x is a binary number and indicates which factorial effect is involved.

For a 3 factor experiment

C^{100} represents main effect of 1st factor

C^{010} represents main effect of 2nd factor

C^{110} represents interaction between 1st and 2nd factors.

For a general case of n factor experiment we write the contrast matrix C^x as

$$C^x = C^{x_1 \ldots x_n} = C_{m_1}^{x_1} \otimes C_{m_2}^{x_2} \otimes \ldots \otimes C_{m_n}^{x_n} \tag{25}$$

where

$$C_{m_j}^{x_j} = \begin{cases} m_j I_{m_j} - J_{m_j} & \text{if} \quad x_j = 1 \\ 1'_m & \text{if} \quad x_j = 0 \end{cases} \qquad (26)$$

A generalised interaction may be represented by a^x .

Kurkjian and Zelan (1962) have shown that the estimates of the generalised interaction a^x is

$$\hat{a}^x = \hat{C}^x \hat{t} \qquad (27)$$

Recall (15) for the GC/n design

$$\hat{t} = WQ$$

Now for factorial experiments we have

$$\hat{a}^x = C^x WQ \qquad (28)$$

The following results follow

(1) Assuming $\hat{a}^x$ is estimable then

$$\text{var}(\hat{a}^x) = C^x W C^{x'} s^2 \qquad (29)$$

(2) As mentioned at the end of introduction GC/n designs have factorial structures.

We have factorial structure if the adjusted treatment sum of squares from the block part of the model can be partitioned orthogonally into main effect and interaction sum of squares.

Where an orthogonal factorial structure is defined to be

$$C^x W C^y = 0 \quad \text{for all} \quad x \neq y$$

Result of Cotter, John and Smith (1973) for presence of factorial structures requires

$$W = \sum_{i_1=1}^{m_1} \cdots \sum_{i_p=1}^{m_p} \left\{ \left(\prod_{j=1}^{p} \otimes R_{m_j}^{i_j} \right) \otimes W_{i_1 \ldots i_p} \right\} \qquad (30)$$

where $W_{i_1 \ldots i_p}$ is a square matrix of order m_n with all column sums and row sums equal.

For GC/n designs, recall (22)

$$W = \sum_{i_1} \cdots \sum_{i_n} w_{i_1 \ldots i_n} \left(\prod_j \otimes R_{m_j}^{i_j} \right)$$

(22) and (30) have equivalent structures if

$$W_{i_1 \ldots i_p} = \sum_{i_n=1}^{m_n} \left(R_{m_n}^{i_n} \otimes w_{i_1 \ldots i_n} \right) \tag{31}$$

Since $W_{i_1 \ldots i_p}$ in a circulant matrix all row sums and column sums are equal and therefore GC/n designs have orthogonal factorial structures.

(3) Supposing orthogonality of factorial structures

$$C^x \, W \, C^y = \begin{cases} C^x \, W & x=y \\ 0 & x \neq y \end{cases} \tag{32}$$

giving $C^x = C^x \, W \, A$ as the estimability condition.

(4) Sum of square due to the hypothesis H_0

where $H_0\colon \; C^x \, t = \underset{\sim}{0}$ is

$$\begin{aligned} S.S(H_0) &= (C^x \, \hat{t})'(C^x \, W \, C^{x'})^-(C^x \, \hat{t}) \\ &= (C^x \, T)'(C^x \, \hat{t}) \end{aligned} \tag{33}$$

T is the vector of unadjusted treatment totals.

t is the vector of treatment effects.

3.2 BALANCE

If a design has balance with respect to all main effects and interactions it is said to have factorial-balance. The analysis

of designs with factorial balance is straight forward. However it is not always possible to construct designs with factorial balance.

We let h to be an eigenvector of W. Then $h't$ is called a BASIC CONTRAST. The importance of basic contrasts is based on the independence of the estimators of their treatment effects. Suppose there are two basic contrasts $h_1't$ and $h_2't$ then if they are orthogonal that is $h_1'h_2=0$, the covariance between estimators $h_1'\hat{t}$ and $h_2'\hat{t}$ is zero and hence they are independent.

For a generalised interaction h can be represented by

$$h = \prod_{j=1}^{n} \otimes h_j^{x_j} \tag{34}$$

where $h_j^{x_j}$ is a $m_j \times 1$ vector

$$\text{and} \quad h_j^{x_j} = \begin{cases} \text{contrast vector if} & x_j=1 \\ \text{vector of 1's if} & x_j=0 \end{cases}$$

Now using (33) we obtain the factorial sum of square for the basic contrasts.

$$H_0: \quad h'\,t = 0$$

$$\begin{aligned} SS(H_0) &= (h'\,\hat{t})'\,(h\,W\,h')^{-}\,(h'\,\hat{t}) \\ &= (h'\,\hat{t})'\,(h'\,\hat{t})/L(h'\,h) \end{aligned} \tag{35}$$

$L(h'\,h)$ is a scalar

But $h'\,\hat{t} = h'\,W\,T$ From (13) and (14)

(Since $t = W\,Q$, with the correction factor zero)

and $h'\,W\,T = L\,h'\,T$

where L is an eigenvalue of W

and h is an eigenvector of W.

Substituting in (35) we have

$$SS(H_0) = \frac{(L\ h'\ T)'\ (L\ h'\ T)}{L(h'\ h)}$$

$$= \frac{L(h'\ T)'\ (h'\ T)}{(h'\ h)} \qquad (36)$$

If none of the eigenvalues of corresponding eigenvectors of W are zero. Then it would be possible to find an orthogonal decomposition of the factorial sum of squares into single degrees of freedom component (34, 36). Given that one or more of the eigenvalues are zero then it shows that the corresponding main effect or interaction is not estimable. However we have already assumed that all main effects and interactions are estimable.

A main affect is balanced if all its corresponding treatment differences can be estimated with the same accuracy.

Now an interaction is balanced if all eigenvalues corresponding to subsets of h are equal. This implies that all linear combinations of the eigenvectors have the same eigenvalue. If all eigenvalues corresponding to a set of eigenvectors spanning the contrast space (for a generalised interaction) are distinct, then any linear combination of these eigenvectors will give a unique set with a distinct eigenvalue.

Analysis for a design with factorial balance is quite simple. The sum of squares can be obtained from $SS(H_0) = L(h'\ T)^2/h'\ h$ and the variance from $S^2\ L(h'\ h)$.

For an S factor interaction we let

$$h = h^* \otimes 1 \qquad (s \leq n) \qquad (37)$$

(without loss of generality we consider the first s factors). And 1 is a vector of size $m_{s+1} \times \ldots \times m_n$

$$h^* = h_1^{x_1} \otimes \ldots \otimes h_s^{x_s} \qquad (x_1 \text{ to } x_s = 1)$$

Then

and

$$h = h_1^{x_1} \otimes \ldots \otimes h_n^{x_n} \qquad (x_{s+1} \text{ to } x_n = 0)$$

John 1979,b has shown that if h is an eigenvector of NN' with eigenvalue L, then h^* is an eigenvector of $N_s N_s'$ with the same eigenvalue. N_s is the incidence matrix of a GC/s design and the initial block is obtained by taking the 1st s elements of the treatment combinations in the initial block of the GC/n design.

<u>EXAMPLE</u> For $n=3$, $s=2$.

An intial block for GC/3 is 000 011 101 210

for GC/2 is 00 01 10 21 .

It is clear that for any GC/n design N_s can be obtained simply by summing certain elements of N. In this case where the interaction of the 1st s factors is considered it is the collapsing of the design matrix from the $(s+1)$ th to n th blocking levels.

For a GC/n design, if a GC/s design has factorial balance then all lower order interactions involving these s factors have also balance. Hence good designs can be built up from balanced GC/1 and GC/2 designs.

3.3 <u>EFFICIENCY</u>

When we are dealing with a comparative study of experimental designs, for a given size there may be a large number of designs to choose from. Therefore we need a criterion to distinguish between the designs and choose a design that suits our demands. A common criterion is known as EFFICIENCY. It is a method of comparing the

variance of possible designs with one standard design. This standard design is taken to be a randomised block design.

$$\text{EFFICIENCY} = \frac{\text{Average variance of estimated treatment contrasts in randomised block.}}{\text{Average variance of estimated treatment contrasts in a particular design.}}$$

It is assumed that error variances s^2 is the same for all designs. The Efficiency factor has always a value less than or equal to 1. The reason is that the randomized block design by involving a full replication of treatments an equal number of times produces a lower bound on the variance.

<u>EXAMPLE</u>

For a pairwise comparison of t_1 and t_2 we have

$$\text{var}(t_1-t_2) = 2\ s^2/r \quad , \quad \text{for a randomised block design.}$$

For some other design we have

$$\begin{aligned}\text{var}(t_1-t_2) &= (w_{11} + w_{22} - 2w_{12})s^2 \\ &= c_{12}\ s^2\end{aligned}$$

where w_{ij} are values of matrix W with $n=1$.

Then the efficiency factor for comparing 1st and 2nd treatments is E_{12}

$$E_{12} = \frac{2\ s^2/r}{c_{12}\ s^2} = \frac{2}{rc_{12}}$$

giving the overal efficiency factor

$$E = \frac{2\ s^2/r}{\bar{c}}$$

where

$$\bar{c} = 2\ s^2 \sum_{i>j}\sum c_{ij}/t(t-1)$$

Efficiency for a factorially balanced design.

For a randomised block design we have $W = 1/r(I)$

For a factorially balanced design variance is $L(h'h)s^2$.

Then using (35) we obtain that efficiency for a factorially balanced design is $(rL)^{-1}$.

Now in a GC/n design with n factors John (1973) has shown the efficiency of the main effect of the ith factor in

$$E(x) = \frac{m_i - 1}{rc} \tag{38}$$

$$c = \sum_{h=2}^{m_i} 1/L_h \tag{39}$$

$$L_h = r - (1/k) \sum_{\ell=1}^{m_i} n_\ell \cos((h-1)(\ell-1) \frac{2\pi}{m_i}) \quad (h=2 \ldots m_i) \tag{40}$$

where n_ℓ are the elements of the 1st row of NN' matrix.

In a generalised interaction we have (recall 29),

$$\mathrm{var}(C^x \hat{t}) = C^x W C^{x'} s^2$$

$W = (1/r)I$ is a randomised block design and hence

$$E(k) = \frac{\mathrm{trace}\ (C^x C^{x'} s^2)}{r\ \mathrm{trace}\ (C^x W C^{x'} \ {}^2} \tag{41}$$

Due to orthogonality of treatment contrasts $C^x W C^{x'} = C^x W$ (Recall 32) giving

$$E(k) = \frac{\mathrm{trace}\ (C^x)}{r\ \mathrm{trace}\ (C^x W)} \tag{42}$$

$$= \frac{\prod_{j=1}^{n} (m_j - 1)^{x_j}}{r\ \mathrm{trace}\ (C^x W)}$$

John (1979,a) has shown that, overall efficiency in GC/n designs can be related to the efficiency of lower order interaction.- The average efficiency for generalised interaction $C^x t$ is

$$E(x) = V_x / \left\{ V_x E_s^{-1} - \sum_{x(s)} V_{x(s)} E^{-1} [x(s)] \right\} \tag{43}$$

where E_s is the overall efficiency for a GC/s design

$E[k(s)]$ is the efficiency of main effects up to s factor interaction depending on the binary value of (s)

$V_k = (m_1-1)(m_2-1)\ldots(m_k-1)$, degree of freedom associated with $C^x t$.

$V_{k(s)}$ in degrees of freedom associated with generalised interaction $C^{x(s)} t$.

Using the above result it can be shown that for $n=1$, since there are no lower order efficiency factors

$$\sum_{x(s)} V_{x(s)} E^{-1} [x(s)] = 0$$

giving

$$E(x) = \frac{(m-1)}{(m-1)E_1^{-1}}$$

Hence overall efficiency in equivalent to the main effect efficiency.

The importance of this result is however based on the way higher order interactions can be evaluated.

Having the main effect efficiencies for an n factor design we can obtain the two factor interactions as follows

$$E(110\ldots0) = (m_1-1)(m_2-1) \Big/ \left[(m_1 m_2 - 1)E_2^{-1} - (m_1-1)E^{-1}(10..0) - (m_2-1)E^{-1}(010\ldots0) \right] \tag{44}$$

This is a very convenient result for factorial experiments. Main affects have highest priority for efficiency and so on for lower order interactions. This result shows that using GC/n designs it is possible to concentrate the effort of design making on obtaining n sets of cyclic designs for the estimation of the main effects and then proceed to higher order interactions. This result is clearly similar to the final result obtained for balance.

EXAMPLE

Consider a design of size (12,4,4; 3,2,2) with the initial block

(000 011 101 210)

For the main effects of the above design we have

E(100) = 0.94 E(010) = 1.00 E(001) = 1.00

The above values for the efficiency of the main effects are satisfactory and now we consider high order interactions

E(011) = 1.00 E(110) = 0.81 E(101) = 0.81 E(111) = 0.44

We obtain high values for 2 factor interactions. The value obtained for the 2nd order interaction is low, however it may be of little interest considering (43) it is clear that possibly we can increase the efficiency of the 2nd order interaction at the expense of the efficiency of 1st order interactions.

4.1 CONSTRUCTION OF GOOD DESIGNS

We have two criteria for distinguishing designs, balance and efficiency. The final results of both efficiency and balance are similar in that with both criteria, designs have to be built for good main effects and then designs with good main effect efficiency or balance can be combined in such a way as to give high efficiency and balance for the 2 factor interactions.

From now on the purpose of this work is to construct a set of cyclic designs for different values of k and m such that a table of relatively efficient designs can be obtained. Once a set of efficient cyclic designs is available, for different values of k every two cyclic designs will be combined to obtain efficient GC/2 designs for different values of m_1 and m_2 .

4.2 CONSTRUCTION OF GC/1 DESIGNS

As was defined previously, size of a GC/n design is represented by $(t,k,r;m_1,\ldots,m_n)$. Number of possible permutations of all possible blocks can be obtained from the above parameters.

Primarily we are concerned with the case of $n=1$. A block size of size k and number of treatments of m gives m^k possible blocks. However since the position of treatments in a block is not important, we consider only the blocks that have their treatments in ascending order. This reduces the total number of blocks to $(m)(k-1)!$.

It is clear that number of blocks obtained in a set of $m(k-1)!$ blocks is very large. Therefore we use a futher method to reduce the number of blocks. Expanding the efficiency formula for the main effects (40)

$$L_h = r - (1/k) \sum n_\ell \cos\left[(h-1)(\ell-1)\theta\right]$$

For $h=2$ writing values of $(h-1)(\ell-1)$ we have

$$
\begin{array}{lllllll}
 & 0 & 1 & 2 & 3 & \cdots & m_i-1 \\
\text{For } h=3 \text{ we have} & 0 & 2 & 4 & 6 & \cdots & 2(m_i-1) \\
\quad\vdots & \vdots & & & & & \\
\text{For } h=m_i \text{ we have} & 0 & (m_i-1) & 2(m_i-1) & & \cdots & (m_i-1)^2
\end{array}
$$

if the above values are greater than m_i we have to take mode m_i so that it is representing one of the treatments. Ignoring the first column of zeros the rest of the above values can be represented by a randomized block allocation of the m_i-1 treatments.

<u>EXAMPLE</u> For $m=5$, $(n-1)(\ell-1)$ values after taking mod m_i can be represented by

h=2	0	1	2	3	4
h=3	0	2	4	1	3
h=4	0	3	1	4	2
h=5	0	4	3	2	1

Our intention is to maximize efficiency for the main effect.

i.e. Minimize C

and Maximize each L_h

and Minimize $n_\ell \cos\left[(n-1)(\ell-1)\theta\right]$

n_1 is always the greatest contributor to values of L_h. n_1 is multiplied by the value of $\cos(0)=1$. While $n_2 \ldots n_m$ are multiples of values less than 1.

For a cyclic representation of NN', the n_1 values are in fact the diagonal elements of this matrix, and in binary designs (see later) they represent the number of times each treatment is replicated.

For a general case

$$n_\ell = \sum_{j=0}^{m-1} f(j)f(j+\ell-1) \tag{45}$$

where $f(j)$ are values of the first column of N

note $j+\ell-1$ is reduced by mod m_i where necessary.

and N is a matrix that represents the number of times each treatment occurs in each block.

EXAMPLE

$k=5$, $m=5$ and initial block 00123 produces the following design

		Block	Treatments 0	1	2	3	4
(00123)		1	2	1	1	1	0
(11234)		2	0	2	1	1	1
(22340)	The matrix N is	3	1	0	2	1	1
(33401)		4	1	1	0	2	1
(44012)		5	1	1	1	0	2

n_1 is equal to multiplication of any column of N by its own transpose. It is minimized if $f(i)$ are equal or as equal as possible. This criterion is that the best designs are the ones that give an as equal as possible replications of treatments within a block, it reduces the total number of blocks to a reasonable amount that can be written in a simple way. This set obtained however, contains many equivalent designs and theory regarding non-isomorphic sets can be used to reduce these designs further, to a few initial blocks.

Designs having an N matrix with values zero and one are called binary designs. One major difference between binary and non-binary designs as far as we are concerned is that in binary designs NN' matrix (concurrence matrix) represents the number of times two treatments occur

together (Recall 18). The values of the concurrence matrix are very important for the efficiency of the designs, and it has been conjectured previously by Williams and Patterson (1978), that the most efficient designs are to be found in the class of designs where the off diagonal elements of the concurrence matrix do not differ by more than one.

EXAMPLE

Consider design of size $k=4$, $m=5$.

BINARY: A binary design for the above size has the initial block (0 1 2 3), giving the first column of N , (1,1,1,1,0). Then the first colum of NN' using (45) is (4,3,3,3,3). It is clear that if a full replication of the design is obtained, off diagonal elements of (4,3,3,3,3) represent the number of times 0 appears with the other treatments.

NON BINARY: For the same size a non binary design has initial block as (0 0 1 2) , giving the first column of N , (2,1,1,0,0) and first column of NN' as (6,3,2,2,3)

The conjecture made by Williams and Patterson (1978) is effective for binary designs. When we are concerned with non binary designs there are many situations where the off diagonal elements do not differ by more than one. However the diagonal elements of the design can be decisive in choosing the best design.

Consider a design of size $k=4$, $m=3$ and initial blocks (0 0 1 2) and (0 0 0 1). The two designs have (6,3,3) and (10,3,3) as the first columns of the NN' matrix respectively.

On considering the expansion of (40) it was shown that the diagonal value of the NN' matrix is very important when choosing a design.

Now apart from conjecture of Williams and Patterson we require that with non binary designs the best designs are the ones that give the lowest diagonal value in the NN' matrix. That is between (0 0 1 2) and (0 0 0 1); (0 0 1 2) gives a higher average efficiency.

Binary designs have $\sum_{i=1}^{m} f(i) = K$

giving

$$rK = n_1 + n_2 + \dots + n_m \tag{46}$$

However the diagonal element is equal to the number of replications (r) and hence

$$r(K-1) = n_2 + \dots + n_m \tag{47}$$

(47) is a useful formula to obtain an optimum design with the Williams and Patterson conjecture.

So far in the consideration of efficiency we have dealt with the structure of NN' . This matrix is a very good guide to come close to an optimum efficiency design. However designs with similar values in the structure of the NN' matrix can give different efficiency values, and it is necessary to obtain the efficiency values. Calculation of the efficiency values for GC/n designs is relatively complex in that information on the structure of these designs is used to speed up the algorithm. For our calculations we use the program EFFICIENCY, (see appendix III).

Input to the program is the size of the design and the initial block. The output is the average efficiency for the factorial effects and the overall efficiency.

W is a generalised inverse of A (Recall 16)

$$A = R\,I - (1/k)\,NN'$$

In the program we let

$$W = \left[r\, I - (1/k)\, NN' + J\right]^{-1} \tag{48}$$

The efficiencys obtained for the efficiency of the best cyclic designs are given in the next chapter. Set of all the non isomorphic designs are given with their first column of NN' matrix. For the best designs the actual efficiency values are also given.

One quick method for obtaining the first column of the NN' matrix (in binary designs) is to produce only the concurrence of the zero treatment.

EXAMPLE

K=3, m=7, initial block (0 1 2) then 3 blocks with treatment zero are (0 1 2)

(6 0 1)

(5 6 0)

Giving the first column of the NN' matrix as (3,2,1,0,0,1,2) .
The above blocks are also given in the tables of the cyclic design.

4.3 CONSTRUCTION OF GC/2 DESIGNS

Once the best designs for a given value of K are known. Different cyclic designs of size K can be combined for values of m_1 and m_2 . Depending on the way two cyclic designs are combined the information regarding the interaction between the two factors can be used. We can have many GC/2 designs when two cyclic designs are combined. However as was explained in the last chapter the value of the overall efficiency of the design is directly proportional to the interaction efficiency.

John and Dean (1975) and Dean and John (1975) have given general mthods for constructing single replicate factorial experiments in generalised cyclic designs for both symmetrical and asymmetrical

arrangements. This method is based on obtaining one or more generators, other treatments within the initial block are obtained by multiplying the generator by integers less than m , and taking mods where necessary.

<u>EXAMPLE</u>

For a 4^2 experiment a 12 generator forms the inital block.

00, 12, 20, 32

where (12)×22 = 24 giving 20

(12)×33 = 36 giving 32

In the case of multireplicate designs, the treatments produced from the generator are in fact the treatments that have to be avoided within the same block. Considering the above initial block, we write other blocks with treatment 00 .

(00 12 20 32)

(32 00 12 20)

(20 32 00 12)

(12 20 32 00)

which is clearly not a good design.

Even if we had a better generator for the main effects, say 11, the initial block and other blocks with 00 treatment are

(00 11 22 33)

(33 00 11 22)

(22 33 00 11)

(11 22 33 00)

and again as far as second order effect is concerned, the values of the NN' matrix are far from the optimal, and hence the overall efficiency

is low. The first column of the NN' matrix is (4,0,0,0,0,4,0,0,0,0,4,0,0,0,0,4) , we have 4, fours for position of 00,11,22,33 and all the other values are zero. The sum of the values of the first column of NN' is 16. A better design is the one that spreads this value more evenly.

A better design has initial block (00 11 23 32) giving

00	11	23	32
33	00	12	21
21	32	00	13
12	23	31	00

Except for 00 none of the treatments within the initial block can be generated from the other treatments. The first column of the NN' matrix is (4,0,0,0,0,1,2,1,0,2,0,2,0,1,2,1) .

In fact the above NN' matrix is very good. Since for the two main effects it has full average efficiency, that is E(01)=E(10)=1 and for the interaction E(11)=0.6279. Giving the overall efficiency of 0.7377.

When considering design of good GC/2 designs we can assume that already we have decided on the value of the maximum average efficiency of the main effects. And so the problem is not only finding the highest overall efficiency for different combinations of two cyclic designs, but also considering other cyclic designs with the same maximum efficiency that can give a relatively high overall efficiency for the GC/2 design.

Two GC/1 designs of size K can yield as many as K^2 different combinations of GC/2 designs. Further for each cyclic design there may also be a few more designs with the same efficiency and hence the total number of designs to consider is very large.

To overcome the problem of reducing the total number of possible designs we make the following observation on the structure of the NN' matrix.

Consider design of size (30,4,4;6,5) .

Cyclic design giving the best efficiency is (0 1 2 3), for both main effects.

We write all the treatments for the effects serially.

Effect A	0 1 2 3 4 5 0 1 2 3 4 5 0 1 2 ...
Effect B	0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 ...

The worst possible way of combining these designs is to take

```
0   1   2   3   4   5   0   1   2   ...
|   |   |   |
0   1   2   3   4   0   1   2   3   ...
```

giving

00	11	22	33
54	00	11	22
43	54	00	11
32	43	54	00

A better design can have the combinations

```
0   1   2   3   4   5   0   1   2   ...

0   1   2   3   4   0   1   2   3   ...
```

giving

00	11	23	32
54	00	12	21
42	53	00	14
33	44	51	00

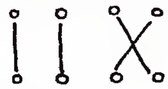
Therefore using the information on the slope of the lines can lead us to a good NN' matrix.

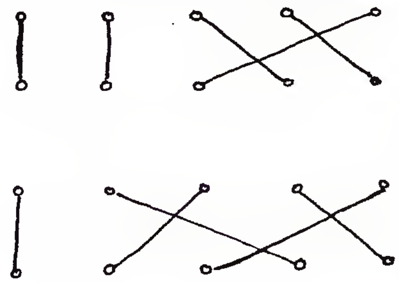
The above method is a relatively easy method of obtaining almost the best designs particularly for the high values of K . When using this method one should be aware of the point that the method although

often leads to very good designs it is dependent only on the value of K .

Consider designs of the following sizes.

(24,4,4;7,4)	0	1	2	3	4	5	6	0	...			
	0	1	2	3	0	1	2	3	...			
(32,4,4;8,4)	0	1	2	3	4	5	6	7	...			
	0	1	2	3	0	1	2	3	...			
(36,4,4;9,4)	0	1	2	3	4	5	6	7	8	0	1	...
	0	1	2	3	0	1	2	3	0	1	2	...

In fact for many designs of size K=4 it was observed that combining the treatments in the form  lead to best designs while for K=5 the best designs are



The method arrives at good designs once the two cyclic designs are known. However it does not provide information for situations where the cyclic design is not decided but the maximum efficiency for the main effects is known. In such cases we obtain a set of good designs.

In the beginning of this section we mentioned the difference for the initial blocks of single replicate and multireplicate designs. That is for multireplicate designs we require that as few as possible of the treatments within an initial block can be generated from each other. This is a good method of reducing the number of designs if we have a set of good designs.

The method is based on obtaining good NN' matrices that have their elements almost or as equal as possible. However there is no reason to suppose that two designs with equivalent forms of NN' matrix should give the same efficiency values. In fact for K=3 differences of order 15% in the efficiency was noted.

Consider designs of size (54,3,3;9,6) and initial blocks (00 11 34),(00 14 31)

(00	11	34)	(00	14	31)
(85	00	23)	(82	00	23)
(62	73	00)	(65	73	00)

The off diagonal elements of NN' matrix of the above designs are either 0 or 1 but the overall efficiency of the first is 0.3128 and for the second is 0.4853.

However the effect of this property is reduced by a great amount such that the difference reduces to order of 5% in the case of K=4 (on the designs that were studied).

So far all the designs we considered were composed of two binary cyclic designs. Now we consider the method for a non binary design.

Size (12,4,4;4,3)   0 1 2 3 0 1 2 3 ...
                     0 1 2 0 1 2 0 1

giving the initial block (00 11 20 32) and overall efficiency of (0.7520).

It was shown that designs of form . . . have low efficiency for all values of K . In fact the intention is to obtain patterns that will give high efficiency and then use the pattern to obtain good designs for other sizes.

Judging a design purely on the basis of obtaining different slopes, may not always give good designs as is shown with the following case.

0 1 2 3 4 5 6 7 ...

0 1 2 3 4 5 6 7 ...

Now considering the NN' matrix we have

00	11	23	35	47
88	00	12	24	36
76	87	00	12	24
64	75	87	00	12
52	63	75	87	00

In fact it is observed that for high values K not only we should choose designs with lowest number of similar slopes but also rate of change should also be considered. Chapter 5 contains tables of some GC/2 designs with relatively high efficiencys.

5.1 Table of cyclic designs.

This table contains information on cyclic designs and their efficiency within the following range.

Block size k, 3 to 8.

Number of treatments m, 2 to 9.

The first column of this table gives the values of k and number of replications r . As a standard r is taken to be equal to k so that the most efficient designs for each block size can be obtained.

The second column gives the number of treatments.

The first two columns refer to the practical constraints of the experiment and these values have to be known.

For situations where there are more than one initial block for value of k and m the 3rd column gives a reference letter to distinguish designs.

By considering the final column of the table for each value of k and m the best initial block cna be obtained. However, the other initial blocks are given for completeness.

The fifth column gives the first column of the NN' matrix. This column is of theoretical interest.

Example

Block size k=4, number of treatments = 9.

The table gives 4 initial blocks.

Designs A and D give the lowest values for the average efficiency and in fact by considering the NN' matrix (5th column) it is clear that these two initial blocks do not provide an optimal allocation of the treatments to the blocks.

Designs B and C, give NN' matrices that have off-diagonal elements differing by 1 only and hence give the highest efficiency.

The most efficient design is C with initial block (0134) and efficiency of 0.8340.

In the above tables it is possible to obtain initial blocks for particular values of m and k in such a way that the diagonal elements of the NN' matrix are higher than the ones given in the tables, but as was discussed this will reduce the efficiency by a great amount.

Example

$k=4$, $m=9$

One possible initial block that is not given in the tables is (0012).

This design gives the value of 6 for the diagonal element of the NN' matrix and hence has a much lower efficiency.

Further, in Appendix I for each initial block all other blocks containing treatment zero are also given.

TABLE 1

Table of all initial blocks of cyclic designs
For Block sizes 3 to 8, number of treatments 2 to 9

	m	Ref	Initial block	1st column of NN'	Efficiency
k=3 r=3	2		001	(5,4)	0.8889
	3		012	(3,3,3)	1.000
	4		012	(3,2,2,2)	0.8889
	5		012	(3,2,1,1,2)	0.8148
	6	A	012	(3,2,1,0,1,2)	0.7435
		B	013	(3,1,1,2,1,1)	0.7843
		C	024	FRACTIONAL	——
	7	A	012	(3,2,1,0,0,1,2)	0.6833
		B	013	(3,2,1,0,0,1,2)	0.7778
	8	A	012	(3,2,1,0,0,0,1,2)	0.6312
		B	013	(3,1,1,1,0,1,1,1)	0.7467
		C	014	(3,1,0,1,2,1,0,1)	0.6914
		D	024	(3,0,2,0,2,0,2)	——
	9	A	012	(3,2,1,0,0,0,0,1,2)	0.5862
		B	013	(3,1,1,1,0,0,1,1,1)	0.7224
		C	036	FRACTIONAL	——
k=4 r=4	2		0011	(8,8)	1.000
	3		0012	(6,5,5)	0.9375
	4		0123	(4,4,4,4)	1.000
	5		0123	(4,3,3,3,3)	0.9375
	6	A	0123	(4,3,2,2,2,3)	0.8937
		B	0124	(4,2,3,2,3,2)	0.8937
		C	0134	FRACTIONAL	——
	7	A	0123	(4,3,2,1,1,2,3)	0.8517
		B	0124	(4,2,2,2,2,2,2)	0.8750
	8	A	0123	(4,3,2,1,0,1,2,3)	0.8095
		B	0124	(4,2,2,1,2,1,2,2)	0.8498
		C	0125	(4,2,1,2,2,2,1,2)	0.8498
		D	0134	(4,2,1,2,2,2,1,2)	0.8498
		E	0145	FRACTIONAL	——
		F	0246	FRACTIONAL	——
	9	A	0123	(4,3,2,1,0,0,1,2,3)	0.7711
		B	0124	(4,2,2,1,1,1,1,2,2)	0.8324
		C	0134	(4,2,1,2,1,1,2,1,2)	0.8340
		D	0136	(4,1,1,3,1,1,3,1,1)	0.8036
k=5 r=5	2		00011	(13,12)	0.9600
	3		00112	(9,8,8)	0.9600
	4		00123	(7,6,6,6)	0.9600
	5		01234	(5,5,5,5,5)	1.000
	6		01234	(5,4,4,4,4,4)	0.9600
	7		01234	(5,4,3,3,3,3,4)	0.9306
	8	A	01234	(5,4,3,2,2,2,3,4)	0.9036
		B	01235	(5,3,3,3,2,3,3,3)	0.9106
		C	01245	(5,3,2,3,4,3,2,3)	0.9081
		D	01246	(5,2,4,2,4,2,4,2)	0.8960
	9	A	01234	(5,4,3,2,1,1,2,3,4)	0.8759
		B	01235	(5,3,3,2,2,2,2,3,2)	0.8958
		C	01236	(5,3,2,3,2,2,3,2,3)	0.8962
		D	01346	(5,2,2,4,2,2,4,2,2)	0.8862

	m	ref	Initial block	1st column of NN'	Efficiency
	2		000111	(18,18)	1.000
	3		001122	(12,12,12)	1.000
	4	A	001123	(10,9,8,9)	0.9623
		B	001223	FRACTIONAL	——
	5		001234	(8,7,7,7,7)	0.9722
k=6	6		012345	(6,6,6,6,6,6)	1.000
r=6	7		012345	(6,5,5,5,5,5,5)	0.9722
	8	A	012345	(6,5,4,4,4,4,4,5)	0.9511
		B	012346	(6,4,5,4,4,4,5,7)	0.9510
	9	A	012345	(6,5,4,3,3,3,3,4,5)	0.9320
		B	012346	(6,4,4,3,3,4,4,4,4)	0.9361
	2		0000111	(25,24)	0.9796
	3		0001122	(17,16,16)	0.9796
	4		0011223	(13,12,12,12)	0.9796
	5		0011234	(11,10,9,9,10)	0.9689
k=7	6		0012345	(9,8,8,8,8,8,8)	0.9796
r=7	7		0123456	(7,7,7,7,7,7,7)	1.000
	8		0123456	(7,6,6,6,6,6,6,6)	0.9796
	9	A	0123456	(7,6,5,5,5,5,5,5,6)	0.9636
		B	0123467	(7,5,5,6,5,5,6,5,5)	0.9635
	2		00001111	(32,32)	1.000
	3		00011122	(22,21,21)	0.9844
	4		00112233	(16,16,16,16)	1.000
	5		00112234	(14,13,12,12,13)	0.9763
k=8	6	A	00112345	(12,11,10,10,10,11)	0.9746
r=8		B	00122345	(12,10,11,10,11,10)	0.9746
		C	00123345	(12,10,10,12,10,10)	0.9740
	7		00123456	(10,9,9,9,9,9,9)	0.9844
	8		01234567	(8,8,8,8,8,8,8,8)	1.000
	9		01234567	(8,7,7,7,7,7,7,7,7)	0.9844

5.2 Tables of GC/2 designs.

These tables contain a set of GC/2 designs for block size k, ranging from 3 to 8, and the number of treatments (m_1 and m_2) for each of the two factors ranging from 2 to 9.

For each set of k, m_1 and m_2, one design is given. This is the design that produces the highest overall efficiency and hence the highest first order interaction efficiency among a large set of designs. Priority was given to the main effect efficiencies and hence the only GC/2 designs that were considered were the ones giving the maximum possible main effect efficiencies (from Table 1).

The last 4 columns of these tables give the efficiency values for the main effects of the first factor, main effect of the 2nd factor, interaction effect efficiency and the overall efficiency, respectively.

Example

For block size 6 and number of treatments 7 and 3, the given design is (00 11 20 32 42 51) giving;

0.9722 and 1.00 as efficiency values for the two main effects,

0.7854 is the interaction effect efficiency and 0.8529 is the overall efficiency of the design.

In the Appendix II for each design in the table 2, k other blocks of the same design that can be produced by a cyclic development and contain the treatment 00 are also given.

TABLE 2A

Table of selected GC/2 designs for block size 3

m_1	m_2	DESIGN	E(10)	E(01)	E(11)	E
2	2	00 11 00	.8889	.8889	.8889	.8889
3	2	00 10 21	1.000	.8889	.5556	.7435
4	2	00 10 21	.8889	.8889	.5333	.6914
5	2	00 10 21	.8148	.8889	.5859	.5998
6	2	00 10 31	.7843	.8889	.4040	.5535
7	2	00 10 31	.7778	.8889	.4866	.6141
8	2	00 10 31	.7467	.8889	.5248	.6292
9	2	00 10 31	.7224	.8889	.5129	.6116
3	3					
4	3	00 11 22	.8889	1.000	.3439	.4820
5	3	00 11 22	.8148	1.000	.2927	.4088
6	3	00 11 32	.7843	1.000	.4796	.5817
7	3	00 11 32	.7778	1.000	.5051	.5975
8	3	00 11 32	.7467	1.000	.4585	.5488
9	3	00 11 32	.7224	1.000	.3895	.4802
4	4	00 12 21	.8889	.8889	.5333	.6349
5	4	00 12 21	.8148	.8889	.4204	.5159
6	4	00 12 31	.7843	.8889	.4912	.5709
7	4	00 11 32	.7778	.8889	.3420	.4238
8	4	00 11 32	.7467	.8889	.4460	.5181
9	4	00 11 32	.7224	.8889	.4674	.5319
5	5	00 12 21	.8148	.8148	.5014	.5752
6	5	00 11 32	.7843	.8148	.4379	.5092
7	5	00 11 32	.7778	.8148	.2991	.3661
8	5	00 11 32	.7467	.8148	.2729	.3336
9	5	00 11 32	.7224	.8148	.3591	.4187
6	6	00 11 34	.7843	.7843	.4724	.5329
7	6	00 11 34	.7778	.7843	.4518	.5094
8	6	00 11 34	.7467	.7843	.3796	.4354
9	6	00 14 31	.7224	.7843	.4359	.4853
7	7	00 13 31	.7778	.7778	.4481	.5012
8	7	00 11 33	.7467	.7778	.2157	.2597
9	7	00 11 33	.7224	.7778	.1974	.2367
8	8	00 13 61	.7467	.7467	.4317	.4763
9	8	00 13 61	.7224	.7467	.4240	.4655
9	9	00 13 61	.7224	.7224	.4176	.4561

TABLE 2B

Table of selcted GC/2 designs for block size 4

m_1	m_2	DESIGN	E(10)	E(01)	E(11)	E
2	2	00 01 10 11	1.000	1.000	1.000	1.000
3	2	00 01 10 21	.9375	1.000	.8125	.8937
4	2	00 10 21 31	1.000	1.000	.6000	.7778
5	2	00 11 21 30	.9375	1.000	.6875	.8119
6	2	00 10 21 31	.8937	1.000	.6224	.7520
7	2	00 10 21 41	.8750	1.000	.6742	.7758
8	2	00 10 21 41	.8498	1.000	.6897	.7737
9	2	00 10 21 41	.8324	1.000	.7052	.7743
3	3	00 01 10 22	.9375	.9375	.7031	.8036
4	3	00 11 20 32	1.000	.9375	.6319	.7520
5	3	00 10 21 32	.9375	.9375	.5996	.7091
6	3	00 10 21 32	.8937	.9375	.6586	.7420
7	3	00 10 21 42	.8750	.9375	.6049	.6937
8	3	00 10 31 42	.8498	.9375	.6698	.7355
9	3	00 10 31 42	.8340	.9375	.6762	.7347
4	4	00 11 23 32	1.000	1.000	.6279	.7377
5	4	00 11 23 32	.9375	1.000	.6875	.7689
6	4	00 11 23 32	.8937	1.000	.6586	.7332
7	4	00 11 23 42	.8750	1.000	.5967	.6746
8	4	00 11 32 43	.8498	1.000	.5666	.6418
9	4	00 11 23 42	.8324	1.000	.6060	.6703
5	5	00 11 23 32	.9375	.9375	.6036	.6849
6	5	00 11 23 32	.8937	.9375	.6561	.7188
7	5	00 11 23 42	.8750	.9375	.6690	.7234
8	5	00 13 32 41	.8498	.9375	.6639	.7133
9	5	00 13 32 41	.8340	.9375	.6575	.7037
6	6	00 11 23 32	.8937	.8937	.6643	.7169
7	6	00 11 23 42	.8750	.8937	.6487	.6984
8	6	00 11 23 42	.8498	.8937	.6539	.6978
9	6	00 13 32 41	.8340	.8937	.6487	.6896
7	7	00 11 24 42	.8750	.8750	.6470	.6921
8	7	00 11 24 42	.8498	.8750	.6470	.6877
9	7	00 11 34 42	.8340	.8750	.6394	.6774
8	8	00 11 34 43	.8498	.8498	.6332	.6712
9	8	00 11 34 42	.8340	.8498	.6491	.6820
9	9	00 11 34 43	.8340	.8340	.6438	.6745

TABLE 2C

Table of selected GC/2 designs for block size 5

m_1	m_2	DESIGN	E(10)	E(01)	E(11)	E
2	2	00 00 01 10 11	.9600	.9600	.9600	.9600
3	2	00 01 10 11 20	.9600	.9600	.9600	.9600
4	2	00 01 10 11 21	.8471	.9600	.7385	.8096
5	2	00 11 20 30 41	1.000	.9600	.7179	.8480
6	2	00 11 20 30 51	.9600	.9600	.7561	.8552
7	2	00 11 20 30 41	.9306	.9600	.7525	.8407
8	2	00 11 20 30 51	.9126	.9600	.7582	.8359
9	2	00 11 20 30 51	.8958	.9600	.7756	.8380
3	3	00 01 10 11 22	.9600	.9600	.8400	.8960
4	3	00 01 10 21 32	.9600	.9600	.7924	.8907
5	3	00 10 21 31 42	1.000	.9600	.6081	.7277
6	3	00 11 21 32 42	.9600	.9600	.7200	.8026
7	3	00 11 21 32 40	.9306	.9600	.7003	.7793
8	3	00 11 21 32 60	.9126	.9600	.7171	.7856
9	3	00 11 21 32 60	.8962	.9600	.7445	.8000
4	4	00 02 11 23 30	.9600	.9600	.7322	.8090
5	4	00 11 20 32 43	1.000	.9600	.7317	.8077
6	4	00 11 20 32 53	.9600	.9600	.7418	.8055
7	4	00 11 20 32 43	.9600	.9600	.7431	.7990
8	4	00 11 23 32 50	.9126	.9600	.7646	.8102
9	4	00 11 23 32 50	.8958	.9600	.7622	.8038
5	5	00 11 23 34 42	1.000	1.000	.7040	.7811
6	5	00 11 24 32 52	.9600	1.000	.7283	.7909
7	5	00 11 23 34 42	.9306	1.000	.7263	.7817
8	5	00 12 21 34 53	.9126	1.000	.7398	.7876
9	5	00 12 21 34 53	.8958	1.000	.7513	.7925
6	6	00 11 23 34 42	.9600	.9600	.7273	.7814
7	6	00 11 22 34 43	.9306	.9600	.7272	.7749
8	6	00 11 23 54 62	.9081	.9600	.7436	.7837
9	6	00 11 23 34 52	.8958	.9600	.7069	.7494
7	7	00 12 21 34 53	.9306	.9306	.7510	.7897
8	7	00 12 21 34 53	.9126	.9306	.7350	.7718
9	7	00 12 21 34 53	.8958	.9306	.7449	.7768
8	8	00 11 23 36 62	.9126	.9126	.7353	.7684
9	8	00 11 23 35 52	.8958	.9126	.7150	.7480
9	9	00 11 23 35 52	.8958	.8958	.7026	.7343

TABLE 2D

Table of selected GC/2 designs for block size 6

m_1	m_2	DESIGN	E(10)	E(01)	E(11)	E
2	2	00 00 01 10 11 11	1.000	1.000	.8889	.9600
3	2	00 01 10 11 20 21	1.000	1.000	1.000	1.000
4	2	00 01 10 11 20 31	.9623	1.000	.9252	.9510
5	2	00 01 10 20 31 41	.9722	1.000	.8432	.9129
6	2	00 10 21 30 41 51	1.000	1.000	.7843	.8889
7	2	00 11 21 30 41 50	.9722	1.000	.7856	.8779
8	2	00 11 21 30 41 60	.9510	1.000	.7723	.8609
9	2	00 11 20 30 41 51	.9320	1.000	.8014	.8689
3	3	00 01 10 12 21 22	1.000	1.000	.8571	.9231
4	3	00 01 10 11 22 32	.9623	1.000	.8400	.8972
5	3	00 01 11 20 32 42	.9722	1.000	.8142	.8783
6	3	00 11 20 32 42 51	1.000	1.000	.7782	.8565
7	3	00 11 20 32 42 51	.9722	1.000	.7854	.8529
8	3	00 11 20 32 42 61	.9510	1.000	.7962	.8537
9	3	00 11 20 32 42 61	.9361	1.000	.7977	.8496
4	4	00 01 11 13 20 32	.9623	.9623	.8247	.8747
5	4	00 01 10 21 32 43	.9722	.9623	.9732	.8497
6	4	00 11 20 32 41 53	1.000	.9623	.7928	.8507
7	4	00 11 23 32 41 50	.9722	.9623	.7955	.8460
8	4	00 11 20 32 51 63	.9511	.9623	.7901	.8365
9	4	00 11 20 32 41 63	.9361	.9623	.8009	.8407
5	5	00 04 11 23 32 40	.9722	.9722	.8078	.8560
6	5	00 11 23 32 40 54	1.000	.9722	.7817	.8358
7	5	00 11 23 32 40 54	.9722	.9722	.7906	.8365
8	5	00 11 23 32 40 54	.9511	.9722	.7889	.8303
9	5	00 11 23 32 40 64	.9361	.9722	.8034	.8385
6	6	00 11 23 32 45 54	1.000	1.000	.7737	.8272
7	6	00 11 23 32 45 54	.9722	1.000	.7757	.8226
8	6	00 11 23 32 45 64	.9510	1.000	.7870	.8270
9	6	00 11 23 32 45 54	.9320	1.000	.7930	.8278
7	7	00 11 23 32 45 54	.9722	.9722	.7690	.8114
8	7	00 11 23 32 45 54	.9511	.9722	.7847	.8203
9	7	00 11 23 32 45 64	.9361	.9722	.7956	.8261
8	8	00 11 23 32 45 54	.9511	.9511	.7666	.8011
9	8	00 11 24 32 45 63	.9361	.9511	.7840	.8130
9	9	00 11 24 32 46 63	.9361	.9361	.8008	.8247

TABLE 2E

Table of selected GC/2 designs for block size 7

m_1	m_2	DESIGN	E(10)	E(01)	E(11)	E
2	2	00 00 01 01 10 10 11	.9796	.9796	.9796	.9796
3	2	00 00 01 10 11 20 21	.9796	.9796	.9796	.9796
4	2	00 01 10 11 20 21 30	.9796	.9796	.9796	.9796
5	2	00 01 10 11 20 30 41	.9689	.9796	.9280	.9514
6	2	00 01 10 20 30 41 51	.9796	.9796	,8643	.9236
7	2	00 10 20 30 41 51 61	1.000	.9796	.7932	.8913
8	2	00 10 20 30 41 51 61	.9796	.9796	.7689	.8685
9	2	00 10 20 30 41 51 71	.9636	.9796	.7963	.8777
3	3	00 01 02 10 11 20 22	.9796	.9796	.9480	.9635
4	3	00 01 10 11 20 22 32	.9796	.9796	.8902	.9286
5	3	00 01 10 11 20 32 42	.9689	.9796	.8585	.9039
6	3	00 01 10 20 31 42 52	.9796	.9796	.8501	.8990
7	3	00 10 20 31 41 52 62	1.000	.9796	.7019	.7956
8	3	00 10 20 31 41 52 62	.9796	.9796	.7095	.7953
9	3	00 10 20 31 41 52 62	.9636	.9796	.7491	.8201
4	4	00 01 10 11 22 23 32	.9796	.9796	.8477	.8960
5	4	00 01 10 11 22 32 43	.9689	.9796	.8214	.8716
6	4	00 01 10 21 32 42 52	.9796	.9796	.8031	.8568
7	4	00 11 23 32 41 52 60	1.000	.9796	.8242	.8737
8	4	00 11 23 32 41 52 60	.9796	.9796	.8230	.8677
9	4	00 11 23 32 41 52 70	.9636	.9796	.8333	.8712
5	5	00 01 10 13 20 34 42	.9689	.9689	.8101	.8569
6	5	00 01 11 23 32 40 54	.9796	.9689	.8273	.8680
7	5	00 11 21 30 43 54 62	1.000	.9689	.7785	.8301
8	5	00 11 21 30 43 54 62	.9796	.9689	.8282	.8651
9	5	00 11 21 30 43 54 72	.9636	.9689	.8322	.8647
6	6	00 01 10 24 32 45 53	.9796	.9796	.7978	.8425
7	6	00 10 21 33 45 52 64	1.000	.9796	.8087	.8506
8	6	00 10 21 33 45 52 64	.9796	.9796	.8268	.8611
9	6	00 12 21 35 43 50 74	.9636	.9796	.8202	.8524
7	7	00 12 21 35 43 56 64	1.000	1.000	.8043	.8457
8	7	00 11 23 32 45 56 64	.9796	1.000	.8218	.8560
9	7	00 11 23 32 45 56 74	.9636	1.000	.8232	.8538
8	8	00 11 23 32 45 56 64	.9796	.9796	.8148	.8464
9	8	00 11 23 32 45 56 74	.9636	.9796	.8224	.8499
9	9	00 11 24 36 45 53 62	.9636	.9636	.8283	.8522

TABLE 2F

Table of selected GC/2 designs for block size 8

m_1	m_2	DESIGN	E(10)	E(01)	E(11)	E
2	2	00 00 01 01 10 10 11 11	1.000	1.000	1.000	1.000
3	2	00 00 01 10 11 11 20 21	.9844	1.000	.9531	.9746
4	2	00 01 10 11 20 21 30 31	1.000	1.000	1.000	1.000
5	2	00 01 10 11 20 21 30 41	.9763	1.000	.9606	.9718
6	2	00 01 10 11 20 30 41 51	.9746	1.000	.9216	.9519
7	2	00 01 10 21 30 41 50 61	.9844	1.000	.8877	.9279
8	2	00 10 20 30 41 51 61 71	1.000	1.000	.8095	.9011
9	2	00 01 10 20 30 41 51 61	.9488	1.000	.8771	.9163
3	3	00 01 02 10 11 12 20 21	.9844	.9844	.9844	.9844
4	3	00 01 10 11 20 22 31 32	1.000	.9844	.9163	.9499
5	3	00 01 10 11 20 21 32 42	.9763	.9844	.9009	.9327
6	3	00 01 10 11 20 31 42 52	.9746	.9844	.8876	.9225
7	3	00 01 10 20 31 41 52 62	.9844	.9844	.8523	.9006
8	3	00 11 21 31 40 52 60 72	1.000	.9844	.8465	.8995
9	3	00 11 21 31 40 52 60 72	.9844	.9844	.8500	.8971
4	4	00 02 11 13 23 20 32 31	1.000	1.000	.8848	.9276
5	4	00 03 11 10 23 21 32 42	.9763	1.000	.8751	.9131
6	4	00 03 11 12 20 31 42 53	.9746	1.000	.8654	.9033
7	4	00 02 11 20 31 42 53 63	.9844	1.000	0.8440	.8875
8	4	00 11 20 31 42 53 63 72	1.000	1.000	.8473	.8912
9	4	00 11 20 31 42 53 63 72	.9844	1.000	.8392	.8810
5	5	00 03 11 12 21 24 30 42	.9763	.9763	.8695	.9024
6	5	00 04 10 11 23 32 41 52	.9746	.9763	.8635	.8953
7	5	00 04 11 21 30 42 53 62	.9844	.9763	.8617	.8937
8	5	00 11 23 32 41 52 64 70	1.000	.9763	.8325	.8719
9	5	00 11 23 32 41 52 64 70	.9844	.9763	.8271	.8642
6	6	00 04 11 15 23 32 40 51	.9746	.9746	.8552	.8862
7	6	00 05 11 23 32 40 51 64	.9844	.9740	.8450	.8774
8	6	00 11 23 32 41 52 64 70	1.000	.9356	.8490	.8773
9	6	00 11 23 32 41 52 64 70	.9844	.9356	.8295	.8591
7	7	00 05 11 23 32 46 50 64	.9844	.9844	.8402	.8721
8	7	00 11 23 32 46 50 64 75	1.000	.9844	.8155	.8514
9	7	00 11 23 32 46 50 64 75	.9844	.9844	.8445	.8725
8	8	00 11 23 32 46 57 64 75	1.000	1.000	.8461	.8761
9	8	00 11 23 32 46 57 64 75	.9844	1.000	.8246	.8550
9	9	00 11 23 32 46 57 64 75	.9844	.9844	.8369	.8627

REFERENCES

BOSE, R.C. and NAIR, K.R. (1939). Partially balanced incomplete block designs. Sankhya 4, 337-372.

BRADLEY, R.A., WALPOLE, R.E. and KRAMER, C.Y. (1960). Intra - and inter-block analysis for factorials in incomplete block designs. Biometrics 16, 566-81.

BRANNALS, L.S. and KRAMER, C.Y. (1961). Factorial treatments in rectangular lattice designs. J. Am. Statis. Ass. 56, 368-78.

COTTER, S.C., JOHN, J.A. and SMITH, T.M.F. (1973). Multi-factor experiments in non orthogonal designs. J. R. Statis. Soc. B. 35, 361-367.

DAVID, H.A. (1963). The structure of cyclic paired comparison designs. J. Aust. Math. Soc. 3, 117-127.

DAVID, H.A. (1965). Enumeration of cyclic paired comparison designs. Ame. Math. Monthly. 72, 241-248.

DAVID, H.A. and WOLOCK, F.W. (1965). Cyclic designs. Ann. Math. Statis. 36, 1526-1534.

DEAN, A.M. and JOHN, J.A. (1975). Single replicate factorial experiments in generalised cyclic designs: II Asymmetrical arrangements. J. R. Statis. Soc. B. 37, 72-76.

FISHER, R.A. (1935). The Design of Experiments, Oliver and Boyd, London.

JOHN, J.A. (1966). Cyclic incomplete block designs. J. R. Statist. Soc. B. 28, 345-360.

JOHN, J.A., WOLOCK, F.W. and DAVID, H.A. (1972). Cyclic Designs. App. Math. Series. No. 62, Washington, D.C. National Bureau of Standards.

JOHN, J.A. (1973). Generalized cyclic designs in factorial experiments. Biometrika 60, 55-63.

JOHN, J.A. and DEAN, A.M. (1975). Single replicate factorial experiments in generalized cyclic designs: I. Symmetrical arrangements. J. R. Statis. Soc. B. 37, 63-71.

JOHN, J.A. and QUENOULLE, M.N. (1977). Experiments: Design and Analysis, Charles Griffin & Company LTD.

JOHN, J.A. (1979 a). On the efficiency factors of generalised cyclic designs in factorial experiments. Unpublished M.S.

JOHN, J.A. (1979 b). Factorial-balance and the analysis of designs with factorial structures. Unpublished M.S.

KEMPTHORNE, O. (1953). A class of experimental designs using blocks of two plots. Ann. Math. Statis. 24, 76-84.

KRAMER, C.Y. and BRADLEY, R.A. (1957). Examples of Intra-block analysis for factorials in group divisible, partially balanced, incomplete block designs. Biometrics 13, 197-224.

KSHIRSAGAR, A.M. (1966). Balanced factorial designs. J. R. Statis. Soc. B. 28, 559-567.

KURKJIAN, B. and ZELEN, M. (1962). A calculus for factorial arrangements. Ann. Math. Statis. 33, 600-19.

KURKJIAN, B. and ZELEN, M. (1963). Applications of the calculus of factorial arrangements. I. Block and Direct product designs Biometrika. 50, 63-73.

MCKEON, J.J. (1960). Some cyclical incomplete paired comparison designs. Tech. Rep. No. 24, Psychometric Lab. of University of North Carolina.

SHAH, B.V. (1958). On balancing in factorial experiments. Ann. Math. Statis. 29, 766-779.

SHAH, B.V. (1960). Balancing factorial experiments. Ann. Math. Statis. 31, 502-514.

SEARLE, S.R. (1970). Linear models, John Wiley and Sons, Inc.

WOLOCK, F.W. (1964). Cyclic designs Ph.D. dissertation virginia polytechnic institute.

YATES, F. (1936). Incomplete Latin squares. J. of Agric. Sci. 26, 301-15.

ZELEN, M. (1958). The use of group divisible designs for confounded asymmetrical factorial arrangements. Ann. Math. Statis. 29, 22-40.

ZOELLNER, J.A. and KEMPTHORNE, O. (1954). Incomplete block designs with blocks of two plots. Res. Bull. No. 418, IOWA, Agric. EXPT. STA.

APPENDIX I

Tables of references for cyclic designs.

In these tables, for each initial block in the table 1, all other blocks of the same designs, that contain the treatment 0 are given.

Each table refers to one particular block size, ranging from 3 to 8. Further, all blocks are classified in the same ordering as the table 1, that is by the number of treatments m and the reference letter.

TABLE 3A

Table of reference for initial blocks of cyclic designs

(block size 3)

m	ref	Block
2		(001) (110)
3		(012)
4		(012) (013) (023)
5		(012) (013) (014) (023) (024) (034)
6	A	(012) (015) (045)
	B	(013) (014) (023) (025) (034) (035)
	C	(024)
7	A	(012) (014) (016))024) (025) (034) (035) (036) (056)
	B	(013) (015) (016) (026) (032) (045)
8	A	(012) (017) (025) (035) (036) (067)
	B	(013) (016) (023) (027) (056) (057)
	C	(014) (015) (034) (037) (045) (047)
	D	(024) (026) (046)
9	A	(012) (015) (018) (024) (027) (035) (048) (057) (078)
	B	(013) (014) (016) (017) (023) (025) (028) (034) (037) (038) (046) (056) (058) (067) (068)
	C	(036)

TABLE 3B

Table of references for initial blocks of cyclic designs

(block size 4)

m	ref	Block
2		(0011)
3		(0012)(0112)(0122)
4		(0123)
5		(0123)(0124)(0134)(0234)
6	A	(0123)(0125)(0145)(0345)
	B	(0124)(0135)(0234)(0243)
	C	(0134)(0235)
7	A	(0123)(0126)(0134)(0135)(0145)(0156)(0235)(0236)(0245)(0248)(0346)(0456)
	B	(0124)(0125)(0136)(0146)(0234)(0256)(0345)(0356)
8	A	(0123)(0127)(0136)(0167)(0235)(0257)(0356)(0567)
	B	(0124)(0126)(0135)(0137)(0146)(0157)(0234)(0236)(0245)(0247) (0256)(0267)(0346)(0357)(0456)(0467)
	C	(0125)(0134)(0147)(0156)(0237)(0345)(0345)(0367)(0457)
	E	(0145)(0347)
	F	(0246)
9	A	(0123)(0128)(0145)(0156)(0178)(0246)(0247)(0257)(0348)(0357) (0458)(0678)
	B	(0124)(0125)(0126)(0127)(0135)(0138)(0148)(0157)(0158)(0168) (0234)(0237)(0245)(0248)(0267)(0278)(0345)(0378)(0456)(0457) (0468)(0478)(0567)(0578)
	C	(0134)(0137)(0146)(0167)(0235)(0238)(0256)(0268)(0347)(0358) (0467)(0568)
	D	(0136)(0147)(0236)(0258)(0345)(0356)(0367)(0368)

TABLE 3C

Table of references for initial blocks of cyclic designs

(block size 5)

m	ref	Block
2		(00011)(11100)
3		(00112)(00122)(01122)
4		(00123)(01123)(01223)(01233)
5		(01234)
6		(01234)(01235)(01245)(01345)(02345)
7		(01234)(01235)(01236)(01245)(01246)
		(01256)(01345)(01346)(01356)(01456)
		(02345)(02346)(02356)(02456)(03456)
8	A	(01234)(01237)(01267)(01346)(01356)
		(01567)(02356)(02357)(02457)(04567)
	B	(01235)(01236)(01247)(01257)(01367)
		(01467)(02345)(02567)(03456)(03567)
	C	(01245)(01256)(01345)(01347)(01456)
		(01457)(02347)(02367)(03457)(03467)
	D	(01246)(01357)(02346)(02456)(02467)
9	A	(01234)(01238)(01256)(01278)(01357)
		(01456)(01458)(01678)(02357)(02457)
		(02467)(02468)(03458)(03478)(05678)
	B	(01235)(01237)(01245)(01246)(01248)(01257)(01267)(01268)(01345)
		(01348)(01358)(01378)(01468)(01567)(01568)(01578)(02345)(02347
		(02348)(02378)(02478)(02456)(02567)(02678)(03457)(03578)(04567)
		(04568)(04578)(04678)
	C	(01236)(01247)(01258)(01356)(01368)(01457)(01478)(02346)(02367)
		(02458)(02578)(03456)(03468)(03567)(03678)
	D	(01346)(01347)(01367)(01467)(02356)(02358)(02368)
		(02568)(03467)(03568)

TABLE 3D

Table of references for initial blocks of cyclic designs

(block size 6)

m	ref	Block
2		(000111)
3		(001122)
4	A	(001123) (001233) (011223) (012233)
	B	(001223) (011233)
5		(001234) (011234) (012234) (012334) (012344)
6		(012345)
7		(012345) (012346) (012356) (012456) (013456) (023456)
8	A	(012345) (012347) (012356) (012367) (012457) (012567)
		(013467) (014567) (023457) (023567) (024567) (034567)
	B	(012346) (012357) (012467) (013567) (023456) (024567)
9	A	(012345) (012348) (012357) (012378) (012456) (012468)
		(012567) (012678) (013458) (013578) (014568) (015678)
		(023457) (023478) (024567) (024678) (034578) (045678)
	B	(012346) (012347) (012356) (012358) (012367) (012368)
		(012457) (012458) (012467) (012478) (012568) (012578)
		(013456) (013457) (013468) (013478) (013567) (013568)
		(013678) (014567) (014578) (014678) (023456) (023458)
		(023467) (023468) (023567) (023578) (023678) (024568)
		(024578) (025678) (034567) (034568) (034678) (035678)

TABLE 3E

Table of references for initial blocks of cyclic designs

(block size 7)

m	ref	Block
2		(0000111) (0001111)
3		(0001122) (0011122) (0011222)
4		(0011223) (0011233) (0012233) (0112233)
5		(0011234) (0012234) (0012334) (0012344)
		(0112234) (0112334) (0112344) (0122334) (0122344) (0123344)
6		(0012345) (0112345) (0122345) (0123345) (0123445) (0123455)
7		(0123456)
8		(0123456) (0123457) (0123467) (0123567) (0124567) (0134567) (0234567)
9	A	(0123456) (0123457) (0123458) (0123468) (0123478) (0123567) (0123578)
		(0123678) (0124567) (0124568) (0124678) (0125678) (0134568) (0134578)
		(0135678) (0145678) (0234567) (0234578) (0234678) (0245678) (0345678)
	B	(0123467) (0123568) (0124578) (0134567) (0134678) (0234568) (0235678)

TABLE 3F

Table of references for initial blocks of cyclic designs

(block size 8)

m	ref	Block
2		(00001111)
3		(00011122) (0001122) (0011122)
4		(00112233)
5		(00112234) (00112334) (00112344) (00122334) (00122344)
		(00123344) (01122334) (01122344) (01123344) (01223344)
6	A	(00112345) (00123455) (01122345) (01223345) (01233445) (01234455)
	B	(00122345) (00123445) (01123345) (01123455) (01223445) (01233455)
	C	(00123345) (01123445) (01223455)
7		(00123456) (01123456) (01223456) (01233456) (01234456) (01234556)
		(01234566)
8		(01234567)
9		(01234567) (01234568) (01234578) (01234678) (01235678) (01245678)
		(01345678) (02345678)

APPENDIX II

Table of reference for GC/2 designs.

For each design in the tables 2A to 2F, in these tables all other blocks of the same designs that can be produced by cyclic development and contain treatment 00 are given.

Each table refers to one particular block size, ranging from 3 to 8. Further all blocks are classified in the same ordering as the tables 2A to 2F, that is by the number of treatments m_1 and m_2 .

TABLE 4A

Table of references for initial blocks of GC/2 designs

(block size 3)

m_1	m_2	Block
2	2	(00 01 10) (00 01 11) (00 10 11)
3	2	(00 10 21) (00 11 20) (00 11 21)
4	2	(00 10 21) (00 11 30) (00 21 31)
5	2	(00 10 21) (00 11 40) (00 31 41)
6	2	(00 10 31) (00 21 50) (00 31 41)
7	2	(00 10 31) (00 21 60) (00 41 51)
8	2	(00 10 31) (00 21 70) (00 51 61)
9	2	(00 10 31) (00 21 80) (00 61 71)
3	3	
4	3	(00 11 22) (00 11 32) (00 21 32)
5	3	(00 11 22) (00 11 42) (00 31 42)
6	3	(00 11 32) (00 21 52) (00 31 42)
7	3	(00 11 32) (00 21 62) (00 41 52)
8	3	(00 11 32) (00 21 72) (00 51 62)
9	3	(00 11 32) (00 21 82) (00 61 72)
4	4	(00 12 21) (00 13 32) (00 23 31)
5	4	(00 12 21) (00 13 42) (00 33 41)
6	4	(00 12 31) (00 23 52) (00 33 41)
7	4	(00 11 32) (00 21 63) (00 42 53)
8	4	(00 11 32) (00 21 73) (00 52 63)
9	4	(00 11 32) (00 21 83) (00 62 73)
5	5	(00 12 21) (00 14 43) (00 34 41)
6	5	(00 11 32) (00 21 54) (00 33 44)
7	5	(00 11 32) (00 21 64) (00 43 54)
8	5	(00 11 32) (00 21 74) (00 53 64)
9	5	(00 11 32) (00 21 84) (00 63 74)
6	6	(00 11 34) (00 23 55) (00 32 43)
7	6	(00 11 34) (00 23 65) (00 42 53)
8	6	(00 11 34) (00 23 75) (00 52 63)
9	6	(00 14 31) (00 23 82) (00 65 73)
7	7	(00 13 31) (00 25 64) (00 46 52)
8	7	(00 11 33) (00 22 76) (00 54 65)
9	7	(00 11 33) (00 22 86) (00 64 75)
8	8	(00 13 61) (00 56 75) (00 27 32)
9	8	(00 13 61) (00 56 85) (00 37 42)
9	9	(00 13 61) (00 57 86) (00 38 42)

TABLE 4B

Table of references for initial blocks of GC/2 designs

(block size 4)

m_1	m_2	Block			
2	2	(00 01 10 11)			
3	2	(00 01 10 21)	(00 01 11 20)	(00 11 20 21)	(00 10 11 21)
4	2	(00 10 21 31)	(00 11 21 30)	(00 10 21 31)	(00 11 21 30)
5	2	(00 11 21 3o)	(00 12 21 41)	(00 11 31 40)	(00 20 31 41)
6	2	(00 10 21 31)	(00 11 21 50)	(00 10 41 51)	(00 31 41 50)
7	2	(00 10 21 41)	(00 11 31 60)	(00 30 51 61)	(00 31 41 50)
8	2	(00 10 21 41)	(00 11 31 70)	(00 20 61 71)	(00 41 51 60)
9	2	(00 10 21 41)	(00 11 31 80)	(00 20 71 81)	(00 51 61 70)
3	3	(00 01 10 22)	(00 02 12 21)	(00 12 20 21)	(00 11 12 21)
4	3	(00 11 20 32)	(00 12 21 32)	(00 12 20 31)	(00 11 22 31)
5	3	(00 10 21 32)	(00 11 22 40)	(00 11 32 42)	(00 21 31 42)
6	3	(00 10 21 32)	(00 11 22 50)	(00 11 42 52)	(00 31 41 52)
7	3	(00 10 21 42)	(60 00 11 32)	(00 21 52 62)	(00 31 41 52)
8	3	(00 10 31 42)	(00 21 32 70)	(00 11 52 62)	(00 41 51 72)
9	3	(00 10 31 42)	(00 21 32 80)	(00 11 62 72)	(00 51 61 82)
4	4	(00 11 23 32)	(00 12 21 33)	(00 13 21 32)	(00 12 23 31)
5	4	(00 11 23 32)	(00 12 21 43)	(00 13 31 42)	(00 22 33 41)
6	4	(00 11 23 32)	(00 12 21 53)	(00 13 41 52)	(00 32 43 51)
7	4	(00 11 23 42)	(00 12 31 63)	(00 23 51 62)	(00 32 43 55)
8	4	(00 11 32 43)	(00 21 32 73)	(00 11 52 63)	(00 41 52 73)
9	4	(00 11 23 42)	(00 12 31 83)	(00 23 71 82)	(00 52 63 71)
5	5	(00 11 23 32)	(00 12 21 44)	(00 14 32 43)	(00 23 34 41)
6	5	(00 11 23 32)	(00 12 21 54)	(00 14 42 53)	(00 23 34 41)
7	5	(00 11 23 42)	(00 12 31 64)	(00 24 52 63)	(00 33 44 51)
8	5	(00 13 32 41)	(00 24 33 72)	(00 14 53 61)	(00 44 52 71)
9	5	(00 13 32 41)	(00 24 33 82)	(00 14 63 71)	(00 54 62 81)
6	6	(00 11 23 32)	(00 12 21 55)	(00 15 43 54)	(00 34 45 51)
7	6	(00 11 23 42)	(00 12 31 65)	(00 25 53 64)	(00 34 45 51)
8	6	(00 11 23 42)	(00 12 31 75)	(00 25 63 74)	(00 44 55 61)
9	6	(00 13 32 41)	(00 25 34 83)	(00 15 64 71)	(00 55 62 81)
7	7	(00 11 24 42)	(00 13 31 66)	(00 25 53 64)	(00 35 46 52)
8	7	(00 11 24 42)	(00 13 31 76)	(00 25 63 74)	(00 45 56 62)
9	7	(00 11 34 42)	(00 23 31 86)	(00 15 63 74)	(00 55 66 82)
8	8	(00 11 34 43)	(00 23 32 77)	(00 17 54 65)	(00 45 56 71)
9	8	(00 11 34 42)	(00 23 31 87)	(00 16 64 75)	(00 56 67 82)
9	9	(00 11 34 43)	(00 23 32 88)	(00 18 65 76)	(00 56 67 81)

TABLE 4C

Table of references for initial blocks of GC/2 designs

(block size 5)

m_1	m_2	Block
2	2	(00 00 01 10 11) (00 01 01 10 11) (00 01 10 10 11) (00 01 10 11 11)
3	2	(00 01 10 11 20) (00 01 10 11 21) (00 01 10 20 21) (00 01 11 20 21) (00 10 11 20 21)
4	2	(00 01 10 11 21) (00 01 10 11 20) (00 01 11 30 31) (00 01 10 30 31) (00 20 21 30 31)
5	2	(00 11 20 30 41) (00 11 21 30 41) (00 10 21 30 41) (00 11 20 31 40) (00 11 20 21 31)
6	2	(00 11 20 30 51) (00 11 21 40 51) (00 10 31 40 51) (00 21 30 41 50) (00 10 21 30 40)
7	2	(00 11 20 30 41) (00 11 21 30 61) (00 10 21 50 61) (00 11 40 51 60) (00 31 40 51 61)
8	2	(00 11 20 30 51) (00 11 21 40 71) (00 10 31 60 71) (00 21 50 61 70) (00 31 40 51 61)
9	2	(00 11 20 30 51) (00 11 21 40 81) (00 10 31 70 81) (00 21 60 71 80) (00 41 50 61 71)
3	3	(00 01 10 11 22) (00 02 10 12 21) (00 01 12 20 21) (00 02 11 20 22) (00 11 12 21 22)
4	3	(00 01 10 21 32) (00 02 12 20 31) (00 11 22 30 31) (00 11 20 22 32) (00 11 12 21 32)
5	3	(00 10 21 31 42) (00 11 21 32 40) (00 13 21 32 42) (00 11 22 32 40) (00 11 21 32 42)
6	3	(00 11 21 32 42) (00 10 21 31 52) (00 11 21 42 50) (00 10 31 42 52) (00 21 32 42 50)
7	3	(00 11 21 32 40) (00 10 21 32 62) (00 11 22 52 60) (00 11 41 52 62) (00 30 41 51 62)
8	3	(00 11 21 32 60) (00 10 21 52 72) (00 11 42 62 70) (00 31 51 62 72) (00 20 31 41 52)
9	3	(00 11 21 32 60) (00 10 21 52 82) (00 11 42 72 80) (00 31 61 72 82) (00 30 41 51 62)
4	4	(00 02 11 23 30) (00 02 13 21 32) (00 12 23 31 33) (00 11 21 23 32) (00 10 12 21 33)
5	4	(00 11 20 32 43) (00 13 21 32 43) (00 12 23 30 41) (00 11 22 33 42) (00 11 22 31 43)
6	4	(00 11 20 32 53) (00 13 21 42 53) (00 12 33 40 51) (00 21 32 43 52) (00 11 22 31 43)
7	4	(00 11 20 32 43) (00 13 21 32 63) (00 12 23 50 61) (00 11 42 53 62) (00 31 42 51 63)
8	4	(00 11 23 32 50) (00 12 21 43 73) (00 13 31 61 72) (00 22 52 63 71) (00 30 41 53 62)
9	4	(00 11 23 32 50) (00 12 21 43 83) (00 13 31 71 82) (00 22 62 73 81) (00 40 51 63 72)

5	5	(00 11 23 34 42) (00 13 21 32 44)	(00 12 23 31 44) (00 13 24 31 42)	(00 11 24 32 43)
6	5	(00 11 24 32 52) (00 20 33 44 51)	(00 13 21 41 54) (00 13 24 32 40)	(00 13 23 41 52)
7	5	(00 11 23 34 42) (00 13 41 52 64)	(00 12 23 31 64) (00 33 44 51 62)	(00 11 24 52 63)
8	5	(00 12 21 34 53) (00 24 51 63 72)	(00 14 22 41 73) (00 42 54 63 71)	(00 13 32 64 71)
9	5	(00 12 21 34 53) (00 24 61 73 82)	(00 14 22 41 83) (00 42 54 63 71)	(00 13 32 74 81)
6	6	(00 11 23 34 42) (00 14 32 43 55)	(00 12 23 31 55) (00 24 35 41 52)	(00 11 25 43 54)
7	6	(00 11 22 34 43) (00 15 42 53 64)	(00 11 23 32 65) (00 33 44 55 61)	(00 12 21 54 65)
8	6	(00 11 23 54 62) (00 14 32 43 55)	(00 12 43 51 75) (00 24 35 41 72)	(00 31 45 63 74)
9	6	(00 11 23 34 52) (00 24 62 73 85)	(00 12 23 41 85) (00 44 55 61 72)	(00 11 25 73 84)
7	7	(00 12 21 34 53) (00 26 43 55 64)	(00 16 22 41 65) (00 34 46 55 61)	(00 13 32 56 61)
8	7	(00 12 21 34 53) (00 26 53 65 74)	(00 16 22 41 75) (00 34 45 55 61)	(00 13 32 66 71)
9	7	(00 12 21 34 53) (00 26 63 75 84)	(00 16 22 41 85) (00 44 56 65 71)	(00 13 32 76 81)
8	8	(00 11 23 36 62) (00 34 52 63 75)	(00 12 25 51 77) (00 26 37 41 54)	(00 13 47 65 76)
9	8	(00 11 23 35 52) (00 25 63 74 86)	(00 12 24 41 87) (00 46 57 61 73)	(00 12 37 75 85)
9	9	(00 11 23 35 52) (00 26 64 75 87)	(00 12 24 41 88) (00 47 58 61 73)	(00 12 38 76 87)

TABLE 4D

Table of references for initial blocks of GC/2 designs

(block size 6)

m_1	m_2	Block		
2	2	(00 00 01 10 11 11)	(00 01 01 10 10 11)	
3	2	(00 01 10 11 20 21)		
4	2	(00 01 10 11 20 31)	(00 01 10 11 21 30)	(00 01 10 21 30 31)
		(00 01 11 20 30 31)	(00 11 20 21 30 31)	(00 10 11 20 21 31)
5	2	(00 01 10 20 31 41)	(00 01 11 21 30 40)	(00 01 10 21 31 40)
		(00 11 21 30 31 40)	(00 10 20 21 31 41)	(00 10 11 21 31 40)
6	2	(00 10 21 30 41 51)	(00 11 20 31 41 50)	(00 11 20 30 41 51)
		(00 11 21 30 40 51)	(00 10 21 31 40 51)	(00 11 21 30 41 50)
7	2	(00 11 21 30 41 50)	(00 10 21 30 41 61)	(00 11 20 31 51 60)
		(00 11 20 40 51 61)	(00 11 31 40 50 61)	(00 20 31 41 50 61)
8	2	(00 11 21 30 41 60)	(00 10 21 30 51 71)	(00 11 20 41 61 70)
		(00 11 30 50 61 71)	(00 21 41 50 60 71)	(00 30 41 51 60 71)
9	2	(00 11 20 30 41 51)	(00 11 21 32 42 81)	(00 10 21 31 70 81)
		(00 11 21 60 71 80)	(00 10 51 60 71 81)	(00 41 50 61 71 80)
3	3	(00 01 10 12 21 22)	(00 02 11 12 20 21)	(00 01 11 12 20 21)
		(00 01 10 12 21 22)	(00 01 10 12 21 22)	(00 02 11 12 20 21)
4	3	(00 01 10 11 22 32)	(00 02 10 12 21 31)	(00 01 12 22 30 31)
		(00 02 11 21 30 32)	(00 10 21 22 31 32)	(00 11 12 21 22 30)
5	3	(00 01 11 20 32 42)	(00 02 10 22 31 41)	(00 12 21 31 40 42)
		(00 12 22 30 31 41)	(00 10 21 22 32 41)	(00 11 12 22 31 40)
6	3	(00 11 20 32 42 51)	(00 12 21 31 40 52)	(00 12 22 31 40 51)
		(00 13 22 31 42 51)	(00 12 21 32 41 50)	(00 12 20 32 41 51)
7	3	(00 11 20 32 42 51)	(00 12 21 31 40 62)	(00 12 22 31 50 61)
		(00 10 22 41 52 61)	(00 10 31 42 51 60)	(00 22 33 42 54 61)
8	3	(00 11 20 32 42 61)	(00 12 21 31 50 72)	(00 12 22 41 60 71)
		(00 10 32 51 62 71)	(00 22 41 52 61 70)	(00 32 40 52 61 71)
9	3	(00 11 20 32 42 61)	(00 12 21 31 50 82)	(00 12 22 41 70 81)
		(00 10 32 61 72 81)	(00 22 51 62 71 80)	(00 42 50 62 71 81)
4	4	(00 01 11 13 20 32)	(00 03 10 12 23 31)	(00 02 13 21 30 33)
		(00 02 11 23 31 32)	(00 12 20 21 31 33)	(00 12 13 21 23 32)
5	4	(00 01 10 21 32 43)	(00 03 13 20 31 42)	(00 11 22 33 40 41)
		(00 11 22 30 33 43)	(00 11 22 23 32 43)	(00 11 12 21 32 43)
6	4	(00 11 20 32 41 53)	(00 13 21 30 42 53)	(00 12 21 33 40 51)
		(00 13 21 32 43 52)	(00 12 23 30 43 51)	(00 11 22 31 43 52)
7	4	(00 11 23 32 41 50)	(00 12 21 30 43 63)	(00 13 22 31 51 62)
		(00 13 22 42 53 61)	(00 13 33 40 52 61)	(00 20 31 43 52 61)
8	4	(00 11 20 32 51 63)	(00 13 21 40 52 73)	(00 12 31 43 60 71)
		(00 23 31 52 63 72)	(00 12 33 44 53 61)	(00 21 32 41 53 72)
9	4	(00 11 20 32 41 63)	(00 13 21 30 52 83)	(00 12 21 43 70 81)
		(00 13 31 62 73 82)	(00 22 53 60 73 81)	(00 41 52 61 73 82)
5	5	(00 04 11 23 32 40)	(00 01 12 24 33 41)	(00 12 21 34 43 44)
		(00 14 22 31 32 43)	(00 13 22 23 34 41)	(00 10 14 21 33 42)

5	5	(00 04 11 23 32 40)	(00 01 12 24 33 41)	(00 12 21 34 43 44)
		(00 14 22 31 32 43)	(00 13 22 23 34 41)	(00 10 14 21 33 42)
6	5	(00 11 23 32 40 54)	(00 12 21 34 43 54)	(00 14 22 31 42 53)
		(00 13 22 33 44 51)	(00 14 20 31 43 52)	(00 11 22 34 43 51)
7	5	(00 11 23 32 40 54)	(00 12 21 34 43 64)	(00 14 22 31 52 63)
		(00 13 22 43 54 61)	(00 14 30 41 53 62)	(00 21 32 44 53 60)
8	5	(00 11 23 32 40 54)	(00 12 21 34 43 74)	(00 14 22 31 62 73)
		(00 13 22 53 64 71)	(00 14 40 51 63 72)	(00 31 42 54 63 71)
9	5	(00 11 23 32 40 64)	(00 12 21 34 53 84)	(00 14 22 41 72 83)
		(00 13 32 63 74 81)	(00 24 50 61 73 82)	(00 31 42 54 63 71)
6	6	(00 11 23 32 45 54)	(00 12 21 34 43 55)	(00 15 22 31 43 54)
		(00 13 22 34 45 51)	(00 15 21 32 44 53)	(00 12 23 35 44 51)
7	6	(00 11 23 32 45 54)	(00 12 21 34 43 65)	(00 15 22 31 53 64)
		(00 13 22 44 55 61)	(00 15 31 42 54 63)	(00 22 33 45 54 61)
8	6	(00 11 23 32 45 64)	(00 12 21 34 53 75)	(00 15 22 41 63 74)
		(00 13 32 54 65 71)	(00 25 41 52 64 73)	(00 22 33 45 54 61)
9	6	(00 11 23 32 45 54)	(00 12 21 34 43 85)	(00 15 22 31 73 84)
		(00 13 22 64 75 81)	(00 15 51 62 74 83)	(00 42 53 65 74 81)
7	7	(00 11 23 32 45 54)	(00 12 21 34 43 66)	(00 16 22 31 54 65)
		(00 13 22 45 56 61)	(00 16 32 43 55 64)	(00 23 34 46 55 61)
8	7	(00 11 23 32 45 54)	(00 12 21 34 43 76)	(00 16 22 31 64 75)
		(00 13 22 55 66 71)	(00 16 42 53 65 74)	(00 33 44 56 65 71)
9	7	(00 11 23 32 45 64)	(00 12 21 34 53 86)	(00 16 22 41 74 85)
		(00 13 32 65 76 81)	(00 26 52 63 75 84)	(00 33 44 56 65 71)
8	8	(00 11 23 32 45 54)	(00 12 21 34 43 77)	(00 17 22 32 65 76)
		(00 13 22 56 67 71)	(00 17 43 54 66 75)	(00 34 45 57 66 71)
9	8	(00 11 24 32 45 63)	(00 13 21 34 52 87)	(00 16 21 47 74 85)
		(00 13 31 66 77 82)	(00 26 53 64 77 85)	(00 35 46 51 67 72)
9	9	(00 11 24 32 46 63)	(00 13 21 35 52 88)	(00 17 22 48 75 86)
		(00 14 31 67 78 82)	(00 26 53 64 77 85)	(00 36 47 51 68 73)

TABLE 4E

Table of references for initial blocks of GC/2 designs

(block size 7)

m_1	m_2	Block
2	2	(00 00 01 01 10 10 11) (00 00 01 01 10 11 11) (00 00 01 10 10 11 11) (00 01 01 10 10 11 11)
3	2	(00 00 01 10 11 20 21) (00 01 01 10 11 20 21) (00 01 10 11 20 20 21) (00 01 10 11 20 21 21) (00 01 10 10 11 20 21) (00 01 11 11 12 21 22)
4	2	(00 01 10 11 20 21 30) (00 01 10 11 20 21 31) (00 01 10 11 20 30 31) (00 01 10 11 21 30 31) (00 01 10 20 21 30 31) (00 01 11 20 21 30 31) (00 10 11 20 21 30 31)
5	2	(00 01 10 11 20 30 41) (00 01 10 11 21 31 40) (00 01 10 20 31 40 41) (00 01 11 21 30 40 41) (00 10 21 30 31 40 41) (00 11 20 21 30 31 40) (00 10 11 20 21 31 41)
6	2	(00 01 10 20 30 41 51) (00 01 11 21 31 40 50) (00 10 20 30 41 50 51) (00 10 21 31 40 41 50) (00 11 21 30 31 40 50) (00 10 20 21 31 41 51) (00 10 11 21 32 41 50)
7	2	(00 10 20 30 41 51 61) (00 10 20 31 41 51 60) (00 10 21 31 41 50 60) (00 11 21 31 40 50 60) (00 10 20 31 41 51 61) (00 10 21 31 41 51 60) (00 11 21 31 41 50 60)
8	2	(00 10 20 30 41 51 61) (00 10 20 31 41 51 70) (00 10 21 31 41 60 70) (00 11 21 31 50 60 70) (00 10 20 41 51 61 71) (00 10 31 41 51 61 70) (00 21 31 41 51 60 70)
9	2	(00 10 20 30 41 51 71) (00 10 20 31 41 61 80) (00 10 21 31 51 70 80) (00 11 21 41 60 70 80) (00 10 30 51 61 71 81) (00 20 41 51 61 71 80) (00 21 31 41 51 60 70)
3	3	(00 01 02 10 11 20 22) (00 01 02 10 12 21 22) (00 01 02 11 12 20 21) (00 01 10 12 20 21 22) (00 02 11 12 20 21 22) (00 02 10 11 12 20 21) (00 01 10 11 12 21 22)
4	3	(00 01 10 11 20 22 32) (00 02 10 12 21 22 31) (00 01 10 12 22 30 31) (00 02 11 12 21 30 32) (00 02 12 20 21 30 31) (00 01 10 21 22 31 32) (00 11 12 21 22 30 31)
5	3	(00 01 10 11 20 32 42) (00 o2 10 12 22 31 41) (00 01 10 22 32 40 41) (00 02 12 21 31 40 42) (00 12 22 30 31 40 41) (00 10 21 22 31 32 41) (00 11 12 21 22 31 40)
6	3	(00 01 10 20 31 42 52) (00 02 12 22 30 41 51) (00 10 21 32 42 50 51) (00 11 22 32 40 41 50) (00 11 21 30 32 42 52) (00 10 21 22 31 41 52) (00 11 12 21 31 42 50)
7	3	(00 10 20 31 41 52 62) (00 10 21 31 42 52 60) (00 11 21 32 42 50 60) (00 10 21 31 42 52 62) (00 11 21 32 42 52 60) (00 10 21 31 41 52 62) (00 11 21 31 42 52 60)
8	3	(00 10 20 31 41 52 62) (00 10 21 31 42 52 70) (00 11 21 32 42 60 70) (00 10 21 31 52 62 72) (00 11 21 42 52 62 70) (00 10 31 41 51 62 72) (00 21 31 41 52 62 70)
9	3	(00 10 20 31 41 52 62) (00 10 21 31 42 52 80) (00 11 21 32 42 70 80) (00 10 21 31 62 72 82) (00 11 21 52 62 72 80) (00 10 41 51 61 72 82) (00 31 41 51 62 72 80)

4	4	(00 01 10 11 22 23 32)	(00 03 10 13 21 22 31)	(00 01 12 13 22 30 31)
		(00 03 11 12 21 30 33)	(00 01 10 22 23 32 33)	(00 03 13 21 22 31 32)
		(00 12 13 22 23 30 31)		
5	4	(00 01 10 11 22 32 43)	(00 03 10 13 21 31 42)	(00 01 12 22 33 40 41)
		(00 03 11 21 32 30 43)	(00 10 21 32 33 42 43)	(00 11 22 23 32 33 40)
		(00 11 12 21 22 33 43)		
6	4	(00 01 10 21 32 42 52)	(00 03 13 20 31 41 51)	(00 11 22 32 42 50 51)
		(00 11 21 31 43 44 53)	(00 10 20 32 33 42 53)	(00 10 22 23 32 43 50)
		(00 12 13 22 33 40 50)		
7	4	(00 11 23 32 41 52 60)	(00 12 21 30 41 53 63)	(00 13 22 33 41 51 62)
		(00 13 20 32 42 53 61)	(00 11 23 33 40 52 61)	(00 12 22 33 41 50 63)
		(00 10 21 33 42 51 62)		
8	4	(00 11 23 32 41 52 60)	(00 12 21 30 41 53 73)	(00 12 33 22 41 61 72)
		(00 13 20 32 52 63 71)	(00 11 23 43 50 62 71)	(00 12 32 43 51 60 73)
		(00 20 31 43 52 61 72)		
9	4	(00 11 23 32 41 52 70)	(00 12 21 34 41 63 83)	(00 13 22 33 51 71 82)
		(00 13 20 42 62 73 81)	(00 11 33 53 60 72 81)	(00 22 42 53 61 70 83)
		(00 20 31 43 52 61 72)		
5	5	(00 01 10 13 20 34 42)	(00 04 12 14 24 33 41)	(00 03 10 24 32 40 41)
		(00 02 12 21 34 42 43)	(00 14 22 30 31 40 43)	(00 13 21 22 31 34 41)
		(00 13 14 21 23 33 42)		
6	5	(00 01 11 23 32 40 54)	(00 04 10 22 31 44 53)	(00 12 21 34 43 50 54)
		(00 14 22 31 42 43 53)	(00 13 22 33 34 44 51)	(00 14 20 21 31 43 52)
		(00 11 12 22 34 43 51)		
7	5	(00 11 21 30 43 54 62)	(00 10 24 32 43 51 64)	(00 14 22 33 41 54 60)
		(00 13 24 32 40 51 61)	(00 11 24 32 43 53 62)	(00 13 21 32 42 51 64)
		(00 13 24 34 43 51 62)		
8	5	(00 11 21 30 43 54 62)	(00 10 24 32 43 51 74)	(00 14 22 33 41 64 70)
		(00 13 24 32 50 61 71)	(00 11 24 42 53 63 72)	(00 13 31 42 52 61 74)
		(00 23 34 44 53 61 72)		
9	5	(00 11 21 30 43 54 72)	(00 10 24 32 43 61 84)	(00 14 22 33 51 74 80)
		(00 13 24 42 60 71 81)	(00 11 34 52 63 73 82)	(00 23 41 52 62 71 84)
		(00 33 44 54 63 71 82)		
6	6	(00 01 10 24 32 45 53)	(00 05 15 23 31 43 52)	(00 14 22 35 43 50 51)
		(00 14 21 35 42 43 52)	(00 13 21 34 35 44 52)	(00 14 21 22 31 45 53)
		(00 13 14 23 31 45 52)		
7	6	(00 10 21 33 45 52 64)	(00 11 23 35 42 54 60)	(00 12 24 31 43 55 65)
		(00 12 25 31 43 53 64)	(00 13 25 31 41 52 64)	(00 12 24 34 45 51 63)
		(00 12 22 33 45 51 64)		
8	6	(00 10 21 33 45 52 64)	(00 11 23 35 42 54 70)	(00 12 24 31 43 65 75)
		(00 12 25 31 53 60 74)	(00 13 25 41 51 62 74)	(00 12 34 44 55 61 73)
		(00 22 32 43 55 61 74)		
9	6	(00 12 21 35 43 50 74)	(00 15 23 31 44 62 84)	(00 14 22 35 53 75 81)
		(00 14 21 45 61 73 82)	(00 13 31 53 63 74 82)	(00 24 40 52 61 75 83)
		(00 22 34 43 51 65 72)		
7	7	(00 12 21 35 43 56 64)	(00 16 23 31 44 52 65)	(00 14 22 35 43 56 68)
		(00 15 21 36 42 54 63)	(00 13 21 34 46 55 62)	(00 15 21 33 42 56 64)
		(00 13 25 34 41 56 62)		
8	7	(00 11 23 32 45 56 64)	(00 12 21 34 45 53 76)	(00 16 22 33 41 64 75)
		(00 13 24 32 55 66 71)	(00 11 26 42 53 65 74)	(00 15 31 42 54 63 76)
		(00 23 34 46 55 61 72)		
9	7	(00 11 23 32 45 56 74)	(00 12 21 34 45 63 86)	(00 16 22 33 51 74 85)
		(00 13 24 42 65 76 81)	(00 11 36 52 63 75 84)	(00 25 41 52 64 73 86)
		(00 23 34 46 55 61 72)		

8	8	(00 11 23 32 45 56 64) (00 12 21 34 45 53 77) (00 17 22 33 41 65 76) (00 13 24 32 56 67 71) (00 11 27 43 54 66 75) (00 16 32 43 55 64 77) (00 24 35 47 56 61 72)
9	8	(00 11 23 32 45 56 74) (00 12 21 34 45 63 87) (00 17 22 33 51 75 86) (00 13 24 42 66 77 81) (00 11 37 53 64 76 85) (00 26 42 53 65 74 87) (00 24 35 47 56 61 72)
9	9	(00 11 24 36 45 53 62) (00 13 25 34 42 51 88) (00 12 21 38 47 75 86) (00 18 26 35 63 74 87) (00 17 26 54 65 78 81) (00 18 46 57 61 73 82) (00 37 48 52 64 73 81)

TABLE 4F

Table of references for initial blocks of GC/2 designs

(block size 8)

m_1	m_2	Block	
2	2	(00 00 01 01 10 10 11 11)	
3	2	(00 00 01 10 11 11 20 21)	(00 01 01 10 10 11 21 20)
		(00 01 01 10 11 20 20 21)	(00 00 01 10 11 20 21 21)
		(00 01 10 10 11 20 21 21)	(00 01 10 11 11 20 20 21)
4	2	(00 01 10 11 20 21 30 31)	
5	2	(00 01 10 11 20 21 30 41)	(00 01 10 11 20 21 31 40)
		(00 01 10 11 20 31 40 41)	(00 01 10 11 21 30 40 41)
		(00 01 10 21 30 31 40 41)	(00 01 11 30 31 40 40 50)
		(00 11 20 21 30 31 40 41)	(00 10 11 20 21 30 31 41)
6	2	(00 01 10 11 20 30 41 51)	(00 01 10 11 21 31 40 50)
		(00 01 10 20 31 41 50 51)	(00 01 11 21 30 40 50 51)
		(00 10 21 31 40 41 50 51)	(00 11 21 30 31 40 41 50)
		(00 10 20 21 30 31 41 51)	(00 10 11 20 21 31 41 50)
7	2	(00 01 10 21 30 41 50 61)	(00 01 11 20 31 40 51 60)
		(00 11 20 31 40 51 60 61)	(00 10 20 31 40 50 51 60)
		(00 11 20 31 40 51 50 61)	(00 11 20 30 31 40 50 61)
		(00 11 20 21 30 41 50 61)	(00 10 11 20 30 41 50 61)
8	2	(00 10 20 30 41 51 61 71)	(00 10 20 31 41 51 61 70)
		(00 10 21 31 41 51 60 70)	(00 11 21 31 41 50 60 70)
		(00 10 20 30 41 51 61 71)	(00 10 20 31 41 51 61 70)
		(00 10 21 31 41 51 60 70)	(00 11 21 31 41 50 60 70)
9	2	(00 01 10 20 30 41 51 61)	(00 01 11 21 31 40 50 60)
		(00 10 20 31 41 51 80 81)	(00 10 21 31 41 70 71 80)
		(00 11 21 31 60 61 70 80)	(00 10 20 51 50 61 71 80)
		(00 10 41 40 51 61 71 80)	(00 31 30 41 51 61 70 80)
3	3	(00 01 02 10 11 12 20 21)	(00 01 02 10 11 12 20 22)
		(00 01 02 10 11 12 21 22)	(00 01 02 10 11 20 21 22)
		(00 01 02 10 12 20 21 22)	(00 01 02 11 12 20 21 22)
		(00 01 10 11 12 20 21 22)	(00 02 10 11 12 20 21 22)
4	3	(00 01 10 11 20 22 31 32)	(00 02 10 12 21 22 30 31)
		(00 01 10 12 21 22 30 31)	(00 02 11 12 20 21 30 32)
		(00 02 11 12 20 21 30 31)	(00 10 12 21 22 31 32 41)
		(00 01 10 12 20 22 31 32)	(00 02 11 12 21 22 30 31)
5	3	(00 01 10 11 20 21 32 42)	(00 02 12 13 22 23 31 41)
		(00 01 10 11 22 32 40 41)	(00 02 10 10 21 31 40 42)
		(00 11 22 22 30 31 40 41)	(00 02 11 21 30 32 42 43)
		(00 10 21 22 31 32 41 42)	(00 11 12 21 22 31 32 40)
6	3	(00 01 10 11 20 31 42 52)	(00 02 10 12 22 30 41 51)
		(00 01 10 21 32 42 50 51)	(00 02 12 20 31 41 50 52)
		(00 11 22 32 40 41 50 51)	(00 11 21 30 32 40 42 52)
		(00 02 10 21 22 31 32 41)	(00 11 12 21 22 31 42 50)
7	3	(00 01 10 20 31 41 52 62)	(00 02 12 22 30 40 51 61)
		(00 10 21 31 42 52 60 61)	(00 11 21 32 42 50 51 60)
		(00 10 21 31 40 42 52 62)	(00 11 21 30 32 42 52 60)
		(00 10 21 22 31 41 52 62)	(00 11 12 21 31 42 52 60)

m_1	m_2	Block	
8	3	(00 11 21 31 40 52 60 72)	(00 10 20 32 41 52 61 72)
		(00 10 22 31 42 51 62 70)	(00 12 21 32 41 52 60 70)
		(00 10 12 32 40 51 61 71)	(00 11 23 31 42 52 62 71)
		(00 12 20 31 41 51 60 72)	(00 11 22 32 42 51 60 71)
9	3	(00 11 21 31 40 52 60 72)	(00 10 20 32 41 52 61 82)
		(00 10 22 31 42 51 72 80)	(00 12 21 32 41 62 70 80)
		(00 12 20 32 50 61 71 81)	(00 11 20 41 52 62 72 81)
		(00 12 30 41 51 61 70 82)	(00 21 32 42 52 61 70 81)
4	4	(00 02 11 13 20 23 31 32)	(00 02 11 13 21 22 30 33)
		(00 02 12 13 20 21 31 33)	(00 02 10 11 22 23 31 33)
		(00 01 12 13 21 23 30 32)	(00 03 11 12 20 22 31 33)
		(00 03 10 12 21 23 31 32)	(00 01 11 13 20 22 32 32)
5	4	(00 03 10 11 21 23 32 42)	(00 01 11 12 20 22 33 43)
		(00 03 10 12 21 31 42 43)	(00 01 11 13 22 32 40 43)
		(00 02 13 23 30 31 41 42)	(00 02 11 21 32 33 40 43)
		(00 10 21 22 32 33 41 43)	(00 11 12 22 23 31 33 44)
6	4	(00 03 11 12 20 31 42 53)	(00 01 12 13 21 32 43 50)
		(00 01 13 20 31 42 52 53)	(00 03 12 23 30 41 51 52)
		(00 11 22 33 40 43 51 52)	(00 11 22 32 33 40 41 53)
		(00 11 21 22 30 33 42 53)	(00 10 11 22 23 31 42 53)
7	4	(00 02 11 20 31 42 53 63)	(00 02 13 22 33 40 51 61)
		(00 13 20 31 42 52 61 62)	(00 11 22 33 43 50 52 61)
		(00 11 22 32 41 43 50 63)	(00 11 21 30 32 43 52 63)
		(00 10 21 23 32 41 51 63)	(00 11 13 22 31 42 53 64)
8	4	(00 11 20 31 42 53 63 72)	(00 13 20 31 42 52 61 73)
		(00 11 22 33 43 52 60 71)	(00 11 22 32 41 53 60 73)
		(00 11 21 30 42 53 62 73)	(00 10 23 31 42 51 62 73)
		(00 13 21 32 41 52 63 70)	(00 12 23 32 43 50 61 71)
9	4	(00 11 20 31 42 53 63 72)	(00 13 20 31 42 52 61 83)
		(00 11 22 33 43 52 70 81)	(00 11 22 32 41 63 70 83)
		(00 11 21 30 52 63 72 83)	(00 10 23 41 52 61 72 83)
		(00 13 31 42 51 62 73 80)	(00 22 33 42 53 64 75 85)
5	5	(00 03 11 12 21 24 30 42)	(00 02 13 14 21 23 32 44)
		(00 01 10 13 24 31 42 44)	(00 04 12 14 23 30 41 43)
		(00 03 14 21 32 34 40 41)	(00 02 11 23 31 34 42 43)
		(00 12 20 23 31 32 41 44)	(00 12 13 20 24 32 34 43)
6	5	(00 04 10 11 23 32 41 52)	(00 01 11 12 24 33 42 53)
		(00 01 13 22 31 42 50 52)	(00 04 12 21 30 41 53 54)
		(00 14 23 34 41 42 52 53)	(00 14 20 32 33 43 44 51)
		(00 11 23 24 30 34 42 51)	(00 12 13 23 24 31 40 54)
7	5	(00 04 11 21 30 42 53 62)	(00 01 12 22 31 43 54 63)
		(00 10 24 31 42 51 63 64)	(00 14 21 32 41 53 54 60)
		(00 12 23 32 40 44 51 61)	(00 11 20 32 33 44 54 63)
		(00 14 21 22 33 43 52 64)	(00 12 13 24 34 43 50 61)
8	5	(00 11 23 32 41 52 64 70)	(00 12 21 30 41 53 64 74)
		(00 14 23 34 41 52 62 73)	(00 14 20 32 43 53 64 72)
		(00 11 23 34 44 55 62 71)	(00 17 23 33 44 56 65 74)
		(00 11 21 32 44 53 62 73)	(00 10 21 33 42 51 62 74)
9	5	(00 11 23 32 41 52 64 70)	(00 12 21 30 41 53 64 84)
		(00 14 23 34 41 52 72 83)	(00 14 20 32 43 63 74 82)
		(00 11 23 34 54 65 72 81)	(00 12 23 43 54 61 70 84)
		(00 11 31 42 54 63 72 83)	(00 20 31 43 52 61 72 84)

m_1	m_2	Block	
6	6	(00 04 11 15 23 32 40 51)	(00 02 11 13 25 34 42 53)
		(00 12 14 21 35 40 53 55)	(00 02 13 23 31 42 51 55)
		(00 15 23 34 43 41 51 54)	(00 14 25 32 34 43 45 51)
		(00 11 20 24 31 35 43 52)	(00 13 15 20 24 32 41 55)
7	6	(00 05 11 23 32 40 51 64)	(00 01 12 24 33 41 52 65)
		(00 12 21 35 40 53 64 65)	(00 15 23 34 41 52 53 64)
		(00 14 25 32 43 44 55 61)	(00 11 24 30 35 41 53 62)
		(00 13 24 25 30 42 51 65)	(00 11 12 23 35 44 52 63)
8	6	(00 11 23 32 41 52 64 70)	(00 12 21 30 41 53 65 75)
		(00 15 24 35 41 53 63 74)	(00 15 20 32 44 54 65 71)
		(00 11 23 35 45 50 62 71)	(00 12 24 34 45 51 60 75)
		(00 12 22 33 45 54 63 74)	(00 10 21 33 42 51 62 74)
9	6	(00 11 23 32 41 52 64 70)	(00 12 21 30 41 53 65 85)
		(00 15 24 35 41 53 73 84)	(00 15 20 32 44 64 75 81)
		(00 11 23 35 55 60 72 81)	(00 12 24 44 55 61 70 85)
		(00 12 32 43 55 64 73 84)	(00 20 31 43 52 61 72 84)
7	7	(00 05 11 23 32 46 50 64)	(00 02 13 25 34 41 52 66)
		(00 12 21 35 46 53 64 66)	(00 16 23 34 41 52 54 65)
		(00 14 25 32 43 45 56 61)	(00 11 25 31 36 42 54 63)
		(00 14 20 25 31 43 52 65)	(00 11 13 24 36 45 52 63)
8	7	(00 11 23 32 46 50 64 75)	(00 12 21 35 46 53 64 76)
		(00 16 23 34 41 52 64 75)	(00 14 25 32 43 55 66 71)
		(00 11 25 36 41 52 64 73)	(00 14 25 30 41 53 62 76)
		(00 11 23 34 46 55 62 73)	(00 12 23 35 44 51 62 76)
9	7	(00 11 23 32 46 50 64 75)	(00 12 21 35 46 53 64 86)
		(00 16 23 34 41 52 74 85)	(00 14 25 32 43 65 76 81)
		(00 11 25 36 51 62 74 83)	(00 14 25 40 51 63 72 86)
		(00 11 33 44 56 65 72 83)	(00 22 33 45 54 61 72 86)
8	8	(00 11 23 32 46 57 64 75)	(00 12 21 35 46 53 64 77)
		(00 17 23 34 41 52 65 76)	(00 14 25 32 43 56 67 71)
		(00 11 26 37 42 53 65 74)	(00 15 26 31 42 54 63 77)
		(00 11 24 35 47 56 62 73)	(00 13 24 35 36 41 52 67)
9	8	(00 11 23 32 46 57 64 75)	(00 12 21 35 46 53 64 87)
		(00 17 23 34 41 52 75 86)	(00 14 25 32 43 66 77 81)
		(00 11 26 37 52 63 75 84)	(00 15 26 41 52 64 73 87)
		(00 11 34 45 57 66 72 83)	(00 23 34 46 55 61 72 87)
9	9	(00 11 23 32 46 57 64 75)	(00 12 21 35 46 53 64 88)
		(00 18 23 34 41 52 76 87)	(00 14 25 32 43 67 78 81)
		(00 11 27 38 53 64 76 85)	(00 16 27 43 53 65 74 88)
		(00 11 35 46 58 67 72 83)	(00 24 35 47 56 61 72 88)

APPENDIX III

COMPUTER PROGRAM EFFICIENCY

This program obtains the average factorial efficiency for each factorial effect, and the overall efficiency.

Input parameters to the program and their limits are:

NF , Number of factors, less than 10

k , block size, less than 10

RDIM , Number of replications

M(I) , Number of levels for each factor, less than 10

IN(I) , the initial block.

The limit on the size of the NN' matrix is 99×99.

Outputs are:

Factorial effects represented in binary form and factorial efficiencies.

ER FORTRAN TEXT

```
      PROGRAM  EFFICIENCY
CCC   CALCULATES THE AVERAGE FACTORIAL EFFICIENCY FOR EACH FACTORIAL EFF
      COMMON NF,V,K,RDIM,NT,M(10),NC(99,99),A(99,99),AMAT(99,99)
      REAL NC
      DIMENSION B(99,99),AA(99,99),BB(99,99),W(99)
      DIMENSION IN(10),MM(10)
      INTEGER RDIM,S,R,C,V
1000     NF=2
       K=8
       RDIM=8
      IF (NF.LE.0) GO TO 999
      WRITE(6,4) NF,K,RDIM
4     FORMAT (28H PARAMETERS N,K,R,M(I) ARE :,3I3)
      READ (5,20) (M(I),I=1,NF)
      WRITE (6,20) (M(I),I=1,NF)
20    FORMAT (10I3)
      N=NF
      V=1
      DO 25 I=1,NF
         V=V*M(I)
25    CONTINUE
      NT=V
      WRITE(6,29)
29    FORMAT (14H INITIAL BLOCK)
      READ (5,30)(IN(I),I=1,K)
      WRITE(6,30)(IN(I),I=1,K)
30    FORMAT (10I7)
      DO 40 I=1,NF
         MM(I)=M(N-I+1)
40    CONTINUE
      DO 100 I=1,K
         LSUM=0.0
      J=NF
         L=0.0
50       IF(J.EQ.1) GO TO 90
         L=IN(I)/(10**(J-1))
         IN(I)=IN(I)-(L*(10**(J-1)))
         LSUM=(LSUM+L)*MM(J-1)
         J=J-1
         GO TO 50
90       IN(I)=LSUM+IN(I)+1
100   CONTINUE
      WRITE (6,200) (IN(I),I=1,K)
200   FORMAT (10I7)
      DO 250 I=1,NT
         NC(1,I)=0.0
250   CONTINUE
      DO 300 I=1,K
         NC(1,IN(I))=NC(1,IN(I))+1
300   CONTINUE
      WRITE (6,350) (NC(1,I),I=1,NT)
350   FORMAT (99F4.0)
      CALL BLOM
      DO 399 I=1,V
         DO 397 J=I,V
```

ER FORTRAN TEXT

```
                AMAT(I,J)=0.0
                    DO 395 L=1,V
                    AMAT(I,J)=NC(L,I)*NC(L,J)+AMAT(I,J)
   395              CONTINUE
                AMAT(I,J)=(-1)*(AMAT(I,J)/K)+1
                AMAT(J,I)=AMAT(I,J)
                IF (I.NE.J) GO TO 397
                AMAT(I,J)=AMAT(I,J)+RDIM
   397          CONTINUE
   399      CONTINUE
            DO 21 I=1,V
            DO 21 J=1,V
            B(I,J) = 0.0
            IF(I.NE.J) GO TO 21
            B(I,J) =1.0
      21    CONTINUE
            IFAIL=1
   CCC
   CCC      INVERT THE V X V MATRIX AMAT. PUT THE INVERSE IN A.
            CALL F04AEF(AMAT,99,B,99,V,V,A,99,W,AA,99,BB,99,IFAIL)
            IF (IFAIL.NE.0) GO TO 1000
   CCC
   CCC
   CCC      CALCULATE THE AVERAGE FACTORIAL EFFICIENCY FOR EACH FACTORIAL EFFE
            CALL FACTEFF
            WRITE(6,15)
      15    FORMAT(1H0)
            GO TO 1000
   999      STOP
            END
   CCC
   CCC
```

BER FORTRAN TEXT

```
      SUBROUTINE BLOM
CCC
CCC   GENERATES A BLOCK CYCLIC MATRIX FROM THE FIRST ROW.
CCC
      COMMON NF,V,K,RDIM,NT,M(10),NC(99,99),A(99,99),AMAT(99,99)
      REAL NC
      INTEGER RDIM,V,S,R,C
      DIMENSION IX(10),IC(10),INC(10)
      M(NF+1) = 1
      INC(NF+1)=1
      NF1=NF-1
      DO 1 I=1,NF
      IX(I)=1
      IC(I)=1
      L=NF-I+1
      INC(L)=M(L)*INC(L+1)
    1 CONTINUE
      IR=1
    4 IND=NF
    5 IIX=IC(IND)
      IIY1=INC(IND+1)
      IIY2=M(IND)
CCC   BUILDS UP A BLOCK CIRCULANT MATRIX USING IIY2 SUBMATRICES.
CCC   IR AND IIX ARE START ROW AND COLUMN POSITIONS IN THE MAIN MATRIX
CCC   IIY1 IS DIMENSION OF EACH SUBMATRIX. ON ENTRY  NC  CONTAINS THE G
CCC   ROWS DEFINING THE PATTERN. ON EXIT IT CONTAINS THE BLOCK CIRCULAN
      CALL CIRC(IR,IIX,IIY1,IIY2,NC)
      IC(IND)=IC(IND)+INC(IND)
      IX(IND)=IX(IND)+1
      IF(IND.EQ.1) GO TO 100
      IF(IX(IND).LE.M(IND-1)) GO TO 4
      IX(IND) = 1
      IND=IND-1
      GO TO 5
  100 RETURN
      END
```

ER FORTRAN TEXT

```
      SUBROUTINE CIRC(IR,IC,M,N,X)
CCC
CCC   PRODUCES A BLOCK CYCLIC PATTERN FROM FIRST GROUP OF  ROWS
CCC   M IS NO. OF ELEMENTS IN EACH ROW OF EACH GROUP.
CCC   N IS NO. OF GROUPS.
CCC
      COMMON NF,V,K,RDIM,NT,MD(10),NCD(99,99),A(99,99),AMAT(99,99)
      REAL NCD
      INTEGER RDIM,S,R,C,V
      DIMENSION X(99,99)
      IRE=IR+(N-2)*M
      ICE=IC+(N-1)*M
      DO 100 IR1=IR,IRE,M
      IR2=IR1+M
      DO 100 IC1=IC,ICE,M
      IC2=IC1+M
      IF(IC1.EQ.ICE) IC2=IC
      JR1 =IR1
      JR2=IR2
      JC1=IC1
      JC2=IC2
      DO 90 I=1,M
      DO 80 J=1,M
      X(JR2,JC2) = X(JR1,JC1)
      JC1 = JC1+1
      JC2=JC2+1
   80 CONTINUE
      JC1=JC1-M
      JC2=JC2-M
      JR1=JR1+1
      JR2=JR2+1
   90 CONTINUE
  100 CONTINUE
      RETURN
      END
```

BER FORTRAN TEXT

```
      SUBROUTINE FACTEFF
CCC
CCC   CALCULATES THE AVERAGE FACTORIAL EFFICIENCY, FEF, FOR EACH EFFECT
CCC   REPRESENTED BY AN NF-TUPLE IX(I), I=1,...,6 AS
CCC                FEF=(PROD*V)/DENOM ,
CCC             WHERE
CCC      PROD=((M(1)**X(1))*...*(M(NF)-1)**X(NF)) ,
CCC      AND DENOM=RDIM*TRACE(X*A) ,
CCC      WHERE X IS THE CONTRAST MATRIX AND A IS A G-INVERSE.
CCC
      COMMON NF,V,K,RDIM,NT,M(10),NC(99,99),A(99,99),AMAT(99,99)
      REAL NC
      INTEGER RDIM,S,R,C,V
      DIMENSION IX(10)
      WRITE(6,14)
   14 FORMAT(1H0,6X,17H FACTORIAL EFFECT,2X,8H AV.EFF.)
      DO 10 I=1,NF
   10 IX(I)=0
      IX(1)=1
      ALL=0.0
   20 IND=1
C
C   IX ARRAY CONTAINS THE NF-TUPLE DEFINING THE FACTORIAL EFFECT
C
      CALL DENOM(IX,B)
C
C     CALCULATES DENOMINATOR OF THE EXPRESSION FOR AV. FACTORIAL EFFECT.
C
      PROD =1.0
      DO 30 J=1,NF
      IF(IX(J).NE.0) PROD=PROD*(M(J)-1.)
   30 CONTINUE
      FEF=PROD*V/B
      ALL=ALL+B
      WRITE(6,15) (IX(J),J=1,NF)
   15 FORMAT(10X,10I1)
      WRITE(6,16) FEF
   16 FORMAT(1H+,22X,F10.4)
C
C     CHANGE IX. ORDER IS 100,010,110,001,101,011,111, FOR NF=3.
C
   40 IX(IND) = IX(IND)+1
      IF(IX(IND).LE.1) GO TO 20
      IF(IND.EQ.NF) GO TO 1000
      IX(IND) = 0
      IND = IND + 1
      GO TO 40
 1000 ALL=(V-1)*(V)/ALL
      WRITE (6,60) ALL
 60   FORMAT(21H  OVERALL EFFICIENCY=,F6.4)
      RETURN
      END
```

BER FORTRAN TEXT

```
      SUBROUTINE DENOM(IX,B)
      COMMON NF,V,K,RDIM,NT,M(10),NC(99,99),A(99,99),AMAT(99,99)
      REAL NC
      INTEGER RDIM,S,R,C,V
      DIMENSION X(99,99),Y(99,99),IX(10),Z(99,99)
C
C     SET UP FIRST ROW OF FIRST MATRIX IN THE KRONECKER PRODUCT.
C
      CALL SMAT(1,IX,X)
      NX=M(1)
      DO 10 I=2,NF
      NY=M(I)
      II=I
      CALL SMAT(II,IX,Y)
C
C     CALCULATE KRONECKER PRODUCT.
C
      CALL KRON (X,Y,Z,NX,NY,NZ,99,99,99)
      NX=NZ
      DO 11 I1=1,NZ
      DO 11 J=1,NZ
      X(I1,J)=Z(I1,J)
   11 CONTINUE
   10 CONTINUE
C
C     CALCULATE TRACE OF PRODUCT OF CONTRAST MATRIX AND G-INVERSE
C
      TRACE=0.0
      DO 15 I=1,V
      DO 15 J=1,V
      TRACE=TRACE+X(I,J)*A(J,I)
   15 CONTINUE
      B=TRACE*RDIM
      RETURN
      END
```

BER FORTRAN TEXT

```
      SUBROUTINE SMAT(II,IX,X)
      COMMON NF,V,K,RDIM,NT,M(10),NC(99,99),A(99,99),AMAT(99,99)
      REAL NC
      INTEGER RDIM,S,R,C,V
      DIMENSION IX(10),X(99,99)
      N=M(II)
      IF(IX(II).EQ.0) GO TO 50
      X(1,1) = M(II) - 1
      DO 10 J=2,N
   10 X(1,J) = -1
      CALL CIRC(1,1,1,N,X)
      RETURN
   50 DO 60 I=1,N
      DO 60 J=1,N
   60 X(I,J) = 1
      RETURN
      END
```

ER FORTRAN TEXT

```
      SUBROUTINE KRON(X,Y,Z,NX,NY,NZ,MX,MY,MZ)
C
C   CALCULATES THE KRONECKER PRODUCT Z =X*Y.
C
      DIMENSION X(MX,99),Y(MY,99),Z(MZ,99)
      IR=1
      IC=1
      DO 200 I=1,NX
      DO 100 J=1,NX
      DO 90 K=1,NY
      DO 80 L=1,NY
      Z(IR,IC) = X(I,J)*Y(K,L)
      IC = IC+1
   80 CONTINUE
      IC=IC-NY
      IR =IR+1
   90 CONTINUE
      IR=IR-NY
      IC=IC+NY
  100 CONTINUE
      IR=IR+NY
      IC=1
  200 CONTINUE
      NZ=NX*NY
      RETURN
      END
```